The Endomorphism Algebras of Jacobians of Some Families of Curves Over Complex Field

HUANG Yong Dong

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Prof. Shing-Tung Yau and Prof.Kang Zuo

摘 要

我們將考慮來自 [4] 中的曲線的族。

f ：$\chi \to S$ 是 P^1 上 $N \geqslant 4$ 個點的泛族，並且 g：$Z \to S$ 是 P^1 的在 χ 上分歧的 d 次迴圈覆蓋的泛族，我們將證明 Kodaira-Spencer 映射是單的，並且對某些族 φ：$S \to {}_{\kappa}M_{\mathbf{C}}$（$G$，$h_0$）是滿的。這裏 ${}_{\kappa}M_{\mathbf{C}}$（$G$，$h_0$）是帶有 Galoise 群作用一個 level 1-結構的主極化 Abel 簇的同構類，所以在這些族中有無限多個複乘點。

我們將決定部分這些族的同源分解，並且將決定這些族的一般元的自同態代數。

Abstract

We will consider the families of curves arising in [4].

Let $f : \chi \longrightarrow S$ be the universal families of $N \geq 4$ points in $\mathbb{P}^1$, and let $g : \mathcal{Z} \to S$ be the families of the d-th cyclic covers of $\mathbb{P}^1$ ramified on χ. We will prove the Kodaira-Spencer map $\mathrm{Kod} : \Theta_S \to R^1 g_* \Theta_{\mathcal{Z}/S}$ is injective and for some of these families the map $\phi : S \to {}_K M_{\mathbb{C}}(G, h_0)$ are dominant, where ${}_K M_{\mathbb{C}}(G, h_0)$ is the moduli space of isomorphism classes of the principally polarized abelian varieties $\mathrm{Jac}(C_\lambda)$ together with the Galois group action and a level 1-structure, so there are infinite many CM points in these families.

We will determine the decomposition of some of these families up to isogenous and determine the endomorphism algebra of a general member of them.

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Contents

Chapter 1

Introduction

We will first review something about abelian varieties and families of abelian varieties with given endomorphism algebras, which are called Shimura varieties.

1.1 Abelian Varieties And Shimura Varieties

An abelian variety X over $\mathbb{C}$ is a complex torus admitting a positive definite line bundle.

If X is an abelian variety, then $\mathrm{End}^0(X)$ (the endomorphism algebra of X over $\mathbb{Q}$) is a finite dimension $\mathbb{Q}$-algebra, any polarization L induces a Rosati-involution $f \to f'$ on $\mathrm{End}^0(X)$, and $(f,g) \to Tr_r(f'g)$ is a positive definite symmetric bilinear form on the $\mathbb{Q}$-vector space $\mathrm{End}^0(X)$.

We say X is simple if X has no abelian subvariety other than 0 and itself. By Poincaŕe's complete reducibility theorem X is isogenous to $X_1^{n_1} \times \ldots \times X_r^{n_r}$, then we have $\mathrm{End}^0(X) \simeq M_{n_1}(F_1) \oplus \ldots \oplus M_{n_r}(F_r)$, where $F_\nu = \mathrm{End}^0(X_\nu)$ are skew fields of finite dimension over $\mathbb{Q}$.

For a simple abelian variety X, the endomorphism algebra of X is a skew field F together with a positive involution $'$, the Rosati involution.

Denote K the center of F, whose fixed field we denote by K_0. Then K_0 is a totally real number field and we have the following theorem about a simple abelian variety.

Theorem 1.1.1 *For pair $(F, ')$ as before, we have the following cases:*

1. $K = K_0$.
   - *(real multiplication)* $F = K$.
   - *(totally indefinite quaternion multiplication) F is a quaternion algebra over K and for every embedding $\sigma : K \hookrightarrow \mathbb{R}$, $F \otimes_\sigma \mathbb{R} \simeq M_2(\mathbb{R})$.*
   - *(totally definite quaternion multiplication) F is a quaternion algebra over K and for every embedding $\sigma : K \hookrightarrow \mathbb{R}$, $F \otimes_\sigma \mathbb{R} \simeq \mathbb{H}$.*
2. *$K \neq K_0$ (complex multiplication). Then K is a totally complex quadratic extension of K_0. Moreover for every embedding $\sigma : K \hookrightarrow \mathbb{C}$, there exists an isomorphism $\varphi : F \otimes_\sigma \mathbb{C} \simeq M_d(\mathbb{C})$.*

But in this paper CM type will be reserved to CM type in the sense of number theory.

Denote $[F : K] = d^2, [K : \mathbb{Q}] = e, [K_0 : \mathbb{Q}] = e_0$ and rank$NS(X) = \rho$. Then we have some restrictions for these values.

Conversely, for every pair $(F, ')$ we can construct families of polarized abelian varieties (X, H) such that $F \hookrightarrow \mathrm{End}^0(X)$. These are called Shimura varieties.

For a general member (X, H, ι) of the Shimura variety associated to $(F, ', \rho)$, we have $\mathrm{End}^0(X) = \iota(F)$ except in the following cases:

1. $(F, ', \rho)$ is of totally definite quaternion type and $m \leq 2$.
2. $(F, ', \rho)$ is of complex multiplication type and $\sum_{\nu=1}^{e_0} r_\nu s_\nu = 0$.
3. $(F, ', \rho)$ is of complex multiplication type and $r_\nu = s_\nu = 1$ for $\nu = 1, \ldots, e_0$.

1.2 Jacobians of Some Families of Curves

Let $g : \mathcal{Z} \to S$ be the following families of curves defined over $\mathbb{C}$

$$\begin{aligned} y^d &= x(x-1)(x-\lambda_1), d > 4 \\ y^d &= x(x-1)(x-\lambda_1)(x-\lambda_2), d = 3, 4, 5, 6, 8, 9, 10, 12, 18 \\ y^d &= x(x-1)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3), d = 3, 4, 6, 8, 12 \\ y^4 &= x(x-1)(x-\lambda_1)\dots(x-\lambda_{N-3}), N = 7, 8. \end{aligned}$$

For simplicity, we will use μ to denote the maximal index of λ.

Let $g : \mathcal{Z} \to S$ be the families as above, the Galois group $G = \mathbb{Z}/d$ acts fibrewise by automorphisms on the families $g : \mathcal{Z} \to S$. We consider the induced families $g : \mathrm{Jac}(\mathcal{Z}/S) \to S$ of Jacobians. Let $\xi = e^{\frac{2\sqrt{-1}\,\pi}{d}}$, then $\mathbb{Z}[t]/(t^{d-1}+\dots+1)$ acts on $\mathrm{Jac}(\mathcal{Z}/S) \to S$ via the action of ξ on $g : \mathcal{Z} \to S$.

Claim 1.2.1 *The ring $\mathbb{Z}[t]/(t^{d-1}+\dots+1)$ acts as a subring of the endomorphism ring of $Jac(\mathcal{Z}/S) \to S$.*

In the proof of this claim, we have got the eigenspaces decomposition of

$$H^1(Z_s, \mathbb{C}) \cong \bigoplus_{i=1}^{d-1} H^1(\mathbb{P}^1 - \{0, 1, \infty, \lambda_1, \dots, \lambda_\mu\}, L_{\xi^i}),$$

where $H^1(L_{\xi^i})$ is the subspace of $H^1(Z_s, \mathbb{C})$ on which σ acts by ξ^i.

The intersection form $<,>$ on the $\mathbb{Q}$-variation of Hodge structures $R^1 g_* \mathbb{Q}_{\mathrm{Jac}(\mathcal{Z})}$ is defined by taking cup product of 1-formes along the fibres of $g : \mathcal{Z} \to S$.

Claim 1.2.2 *For $l \in \mathbb{Z}[t]/(t^{d-1}+\dots+1)$ and for all $u, v \in R^1 g_* \mathbb{Q}_{Jac(\mathcal{Z})} |_{S_0} = H^1(g^{-1}(s_0), \mathbb{Q})$ one has $< lu, v > = < u, \bar{l}v >$.*

The Kodaira-Spencer map of $g : \mathcal{Z} \longrightarrow S$ is

$$\mathrm{Kod} : \Theta_S \longrightarrow R^1 g_* \Theta_{\mathcal{Z}/S}.$$

Claim 1.2.3 *The Kodaira-Spencer map is injective.*

For a suitable choice of K, because of the above claim the family of Jacobians induces a generically finite morphism

$$\phi : S \longrightarrow {}_K M_{\mathbb{C}}(G, h_0),$$

is dominant for the following families

$$\begin{aligned} y^d &= x(x-1)(x-\lambda_1), d = 4,5,6,7,9 \\ y^d &= x(x-1)(x-\lambda_1)(x-\lambda_2), d = 3,4,5 \\ y^3 &= x(x-1)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3), \\ y^2 &= x(x-1)\cdots(x-\lambda_\mu), \mu = 3,4. \end{aligned}$$

Theorem 1.2.1 *There are infinite many CM type points in the following families of Jacobians :*

$$\begin{aligned} y^d &= x(x-1)(x-\lambda_1), d = 4,5,6,7,9 \\ y^d &= x(x-1)(x-\lambda_1)(x-\lambda_2), d = 3,5 \\ y^d &= x(x-1)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3), d = 2,3 \end{aligned}$$

1.3 Endomorphism Algebras of Jacobians of Curves

In this section we will determine the endomorphism algebra of a general member of some families of Jacobians.

Let $f : C \to C'$ be a finite morphism between two smooth projective curves, we have

Proposition 1.3.1 *The homomorphism $f^* : J' \to J$ is not injective if and only* *if f factorizes via a cyclic étale covering f' of degree $n \geq 2$:*

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "f''"'] & C' \\ C'' \arrow[r, equal] & C'' \arrow[u, "f'"'] \end{tikzcd}$$

From the proof of the above proposition we can obtain

Corollary 1.3.1 *For any finite morphism $f : C \to C'$ of smooth projective curves C and C' there is a factorization*

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "g"'] & C' \\ C_e \arrow[r, equal] & C_e \arrow[u, "f_e"'] \end{tikzcd}$$

with f_e étale, $kerf^* = kerf_e^*$, and $g^* : J(C_e) \to J$ injective.

Now we discuss the endomorphism algebra of a general member of the families $\mathrm{Jac}(\mathcal{Z}/S)$. We have the following results.

Claim 1.3.1 *The general member of Jacobians of the family of $y^5 = x(x-1)(x-\lambda_1)$ is simple, and the endomorphism algebra of the general member is $\mathbb{Q}(\xi)$, where ξ is a primitive root of $t^5 - 1 = 0$.*

Claim 1.3.2 *The Jacobians of the family $y^4 = x(x-1)(x-\lambda_1)$ is isogenous to $Jac(y^2 = x(x-1)(x-\lambda_1)) \times E_1^2$, where E_1 is simple abelian variety of CM type of dimension 1. So the endomorphism algebra of the general member of this family is $\mathbb{Q} \oplus M_2(\mathbb{Q}(\sqrt{-1}))$.*

Claim 1.3.3 *The Jacobians of the family $y^6 = x(x-1)(x-\lambda_1)$ is isogenous to $Jac(y^2 = x(x-1)(x-\lambda_1)) \times E_1^3$, where E_1 is a simple abelian variety of CM type of dimension 1 with endomorphism algebra $\mathbb{Q}(\xi)$, where ξ is a primitive root of $t^3 - 1 = 0$.*

Claim 1.3.4 *The general member of Jacobians of the family $y^3 = x(x-1)(x-\lambda_1)(x-\lambda_2)$ is simple, and the endomorphism algebra of the general member is $\mathbb{Q}(\xi)$, where ξ is a primitive root of $t^3 - 1 = 0$.*

Claim 1.3.5 *The family of Jacobians of the normalization of $y^3 = x^2(x-1)(x-\lambda_1)$ is isomorphic to a product of family of elliptic curves $\mathcal{E}_1^2$, the endomorphism algebra of a general member of $\mathcal{E}_1$ is $\mathbb{Q}$.*

Claim 1.3.6 *The Jacobians of the family $y^6 = x(x-1)(x-\lambda_1)(x-\lambda_2)$ is isogenous to $Jac(y^2 = x(x-1)(x-\lambda_1)) \times Jac(y^3 = x(x-1)(x-\lambda_1)(x-\lambda_2)) \times (E_1)^3$, where E_1 is a simple abelian variety of CM type of dimension 1.*

Claim 1.3.7 *The general member of Jacobians of the family $y^2 = x(x-1)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$ is simple, and the endomorphism algebra of the general member is $\mathbb{Q}$.*

Claim 1.3.8 *The general member of the family $y^3 = x(x-1)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$ is simple, and the endomorphism algebra of the general member is $\mathbb{Q}(\xi)$, where ξ is a primitive root of $t^3 - 1 = 0$.*

Chapter 2

Families of Abelian Varieties

In this chapter we will discuss abelian varieties and endomorphisms of abelian varieties and Shimura varieties. Here we follow [9].

2.1 Abelian Varieties

An abelian variety X is a complete algebraic variety over k with a group law $m : X \times X \to X$ such that m and the inverse map are both morphism of varieties. We will only consider abelian varieties over $\mathbb{C}$, then the underlying complex analytic space of an abelian variety is a compact complex analytic group, hence it is a compact complex torus. Then $X = \mathbb{C}^g/\Lambda$ is a complex torus (here Λ is a lattice, i.e. a discrete subgroup of maximal rank). So by definition an abelian variety over $\mathbb{C}$ is a complex torus admitting a positive definite line bundle.

Denote $\mathrm{Hom}(X, Y)$ the group of homomorphisms from X to Y, $\mathrm{End}(X)$ the endomorphism ring of X, and $\mathrm{End}^0(X) = \mathrm{End}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$.

Let $X = V/\Lambda$ be a complex torus of dimension g. Choose a basis $e_1, \dots, e_g$ of V and $\lambda_1, \dots, \lambda_{2g}$ of Λ and let Π be the corresponding period matrix. With respect to these bases we have

$$X = \mathbb{C}^g/\Pi\mathbb{Z}^{2g}$$

Theorem 2.1.1 *X is an abelian variety if and only if there is a nondegenerate alternating matrix $A \in M_{2g}(\mathbb{Z})$ such that i) $\Pi A^{-1}\Pi = 0$, ii) $i\Pi A^{-1}\bar{\Pi} > 0$.*

These conditions are called Riemann Relations. In fact A is the matrix of the alternating form defining the polarization. Denote the alternating form by E on Λ, E can be extended to $\Lambda \bigotimes \mathbb{R} = \mathbb{C}^g$. Define $H : \mathbb{C}^g \times \mathbb{C}^g \to \mathbb{C}$ by

$$H(u, v) = E(iu, v) + iE(u, v)$$

Then the condition i) is equivalent to H is Hermitian, and ii) is equivalent to H is positive definite.

If X is an abelian variety, then $\text{End}^0(X)$ is a finite dimension $\mathbb{Q}$-algebra, any polarization L induces the Rosati-involution $f \to f'$ on $\text{End}^0(X)$. It is the adjoint operator with respect to the hermitian form $H = c_1(L)$.

Theorem 2.1.2 *$(f, g) \to Tr_r(f'g)$ is a positive definite symmetric bilinear form on the $\mathbb{Q}$-vector space $End^0(X)$.*

We say X is simple if X has no abelian subvariety other than 0 and itself. For any abelian variety, we have the following reducibility theorem.

Theorem 2.1.3 (Poincaŕe's Complete Reducibility Theorem) *Given an abelian variety X, there is an isogeny*

$$X \to X_1^{n_1} \times \ldots \times X_r^{n_r}$$

with simple abelian varieties X_ν not isogenous to each other. Moreover the abelian varieties X_ν and the integers n_ν are uniquely determined up to isogenous and permutations.

Corollary 2.1.1 *$End^0(X)$ is a semi-simple $\mathbb{Q}$-algebra. To be more precise: if $X \to X_1^{n_1} \times \ldots \times X_r^{n_r}$ is isogeny as in the previous theorem, then*

$$End^0(X) \simeq M_{n_1}(F_1) \oplus \ldots \oplus M_{n_r}(F_r),$$

where $F_\nu = End^0(X_\nu)$ are skew fields of finite dimension over $\mathbb{Q}$.

2.2 The Endomorphism Algebra of A Simple Abelian Varieties

For a simple abelian variety X, the endomorphism algebra of X is a skew field F together with a positive involution $'$, the Rosati involution.

Denote $(F,\ ')$ a skew field of finite dimension over $\mathbb{Q}$ with positive anti-involution $x \mapsto x'$. The anti-involution $x \mapsto x'$ restricts to an involution on the center K of F, whose fixed field we denote by K_0.

Lemma 2.2.1 *K_0 is a totally real number field, i.e. every embedding $K_0 \hookrightarrow \mathbb{C}$ factorizes via $\mathbb{R}$.*

We have the following theorem about a simple abelian variety.

Theorem 2.2.1 *For pair $(F,\ ')$ as before, F is a skew field of finite dimension over $\mathbb{Q}$ and $'$ positive anti-involution, K is the center of F, K_0 is the fixed field of K with respect to $'$. Then we have the following cases:*

1) $K = K_0$ (the first kind). Then K is a totally real number field and one of the following cases holds.

a) (real multiplication) $F = K$ and $x' = x$ for all $x \in F$.

b) (totally indefinite quaternion multiplication) F is a quaternion algebra over K and for every embedding $\sigma : K \hookrightarrow \mathbb{R}$

$$F \otimes_\sigma \mathbb{R} \simeq M_2(\mathbb{R}).$$

Moreover there is an element $a \in F$ with $a^2 \in K$ totally negative such that the anti-involution $x \mapsto x'$ is given by $x' = a^{-1}\overline{x}a$.

c) (totally definite quaternion multiplication) F is a totally definite quaternion algebra over K. This means F is a quaternion algebra over K and for every embedding $\sigma : K \hookrightarrow \mathbb{R}$

$$F \otimes_\sigma \mathbb{R} \simeq \mathbb{H}.$$

Moreover the anti-involution $x \mapsto x'$ is given by $x' = \overline{x}$.

2) $K \neq K_0$ (the second kind). Then K is a totally complex quadratic extension of K_0, K_0 is a totally real number field. Moreover for every embedding $\sigma : K \hookrightarrow \mathbb{C}$, there exists an isomorphism

$$\varphi : F \otimes_\sigma \mathbb{C} \simeq M_d(\mathbb{C})$$

such that extends via φ to the canonical anti-involution

$$X' = {}^t\overline{X}$$

on $M_d(\mathbb{C})$.

Any other positive anti-involution on F is of the form

$$x \mapsto a x' a^{-1}$$

with $a \in F, a' = a$ and such that $\varphi(a \otimes 1)$ is a positive definite hermitian matrix in $M_d(\mathbb{C})$ for every embedding $\sigma : K \hookrightarrow \mathbb{C}$.

The last case is called the complex multiplication (CM) type in [9], in this paper we will call this the general CM type, the name CM type will be reserved to CM type in the sense of number theory.

Denote

$$[F : K] = d^2, [K : \mathbb{Q}] = e, [K_0 : \mathbb{Q}] = e_0 \text{ and } \mathrm{rank} NS(X) = \rho.$$

Then

Proposition 2.2.1 *We have the following restrictions for these values:*

1) $F = End^0(X)$ is a totally real number field, then $d = 1, e_0 = e, \rho = e, e \mid g$;

2) $F = End^0(X)$ is totally indefinite quaternion algebra, then $d = 2, e_0 = e, \rho = 3e, 2e \mid g$;

3) $F = End^0(X)$ is totally definite quaternion algebra, then $d = 2, e_0 = e, \rho = e, 2e \mid g$;

4) $(F, ')$ is of the second kind, then $e_0 = \frac{1}{2}e, \rho = e_0 d^2, e_0 d^2 \mid g$.

2.3 Family of Abelian Varieties and Shimura Varieties

Now we consider the converse question: which of such pair $(F, ')$ actually occur as the endomorphism algebra of a polarized abelian variety? To be more precise, for every pair $(F, ')$ we construct families of polarized abelian varieties (X, H) together with an endomorphism structure. Roughly speaking, there is an embedding $F \hookrightarrow \mathrm{End}^0(X)$.

First we will give some definitions.

A polarized abelian variety with endomorphism structure $(f, ', \rho)$ is by definition a triplet (X, H, ι) with an abelian variety $X = \mathbb{C}^g/\Lambda$, a positive definite hermitian form H on $\mathbb{C}^g$ defining a polarization on X and an embedding $\iota : F \hookrightarrow \mathrm{End}^0(X) \subseteq M_g(\mathbb{C})$ (here we consider $\mathrm{End}^0(X)$ as a subspace of $M_g(\mathbb{C})$ via the analytic representation) such that

a) ι and ρ are equivalent representation, and

b) the Rosati involution on $\mathrm{End}^0(X)$ with respect to H extends the anti-involution $'$ on F via ι.

An isomorphism $f : (\widetilde{X}, \widetilde{H}, \widetilde{\iota}) \to (X, H, \iota)$ is by definition an isomorphism of of polarized abelian varieties $f : (\widetilde{X}, \widetilde{H}) \to (X, H)$ such that the diagram

$$\begin{array}{ccc} \widetilde{X} & \xrightarrow{f} & X \\ {\scriptstyle \widetilde{\iota}(a)}\downarrow & & \downarrow{\scriptstyle \iota(a)} \\ \widetilde{X} & \xrightarrow{f} & X \end{array}$$

commutes foe all $a \in F$.

The representation $\rho : F \to M_g(\mathbb{C})$ cannot be arbitrary in order to ensure the existence of a polarized abelian variety of type $(F, ', \rho)$. In fact we have the following lemma.

Lemma 2.3.1 *Let $\sigma_1, \ldots, \sigma_e$ denote the irreducible $\mathbb{C}$-representations of $F \otimes_{\mathbb{Q}} \mathbb{C}$. For any rational representation $\psi : F \to M_{2g}(\mathbb{C})$ there is an integer $m \geq 1$ such* *that*

$$\psi \otimes I_{\mathbb{C}} \simeq m \sum_{\nu=1}^{e} \sigma_\nu.$$

We will construct families of polarized abelian varieties to each type endomorphism structure.

2.3.1 Real Multiplication

First we will consider the abelian varieties with real multiplication. Let F be a totally real number field of degree e over $\mathbb{Q}$. We say that an abelian variety X admits real multiplication by F, if there is an embedding $F \hookrightarrow \mathrm{End}^0(X)$. Proposition 2.2.1 gives

$$g = em$$

for some integer $m \geq 1$. The e different embeddings $F \hookrightarrow \mathbb{R}$ determine an isomorphism of $F \otimes_{\mathbb{Q}} \mathbb{R}$ with $\mathbb{R}^e$. Identifying both sides, we may write any $a \in F \otimes_{\mathbb{Q}} \mathbb{R}$ in the form

$$a = \begin{pmatrix} a^1 \\ \vdots \\ a^e \end{pmatrix}$$

Similarly we write the elements of $(F \otimes_{\mathbb{Q}} \mathbb{R})^{2m} = (\mathbb{R}^{2m})^e = \mathbb{R}^{2g}$ as

$$\underline{a} = \begin{pmatrix} \underline{a}^1 \\ \vdots \\ \underline{a}^e \end{pmatrix} \text{ with } \underline{a}^\nu = \begin{pmatrix} a_1^\nu \\ \vdots \\ a_{2m}^\nu \end{pmatrix} \in \mathbb{R}^{2m} \text{ for } 1 \leq \nu \leq e.$$

Define a representation $\rho : F \to M_g(\mathbb{C})$ by

$$\rho(a) = \mathrm{diag}(a^1 \mathbb{I}_m, \ldots, a^e \otimes \mathbb{I}_m) = \mathrm{diag}(a^1, \ldots, a^e) \otimes \mathbb{I}_m.$$

Obviously every representation $F \to M_g(\mathbb{C})$, satisfying the conditions of lemma 2.3.1, is equivalent to ρ. So it suffices to consider the representation ρ.

For any $Z \in \mathfrak{H}_m^e$, where $\mathfrak{H}_m^e$ is the e-fold product of the Siegel upper half space, we will construct a polarized abelian variety (X_Z, H_Z, ι_Z) with endomorphism structure (F, id_F, ρ): fix a free $\mathbb{Z}$-submodule $\mathcal{M}$ of F^{2m} of rank $2g = 2em$ such that

$$\mathrm{tr}_{F/\mathbb{Q}} \left({}^t\underline{a} \begin{pmatrix} 0 & \mathbb{I}_m \\ -\mathbb{I}_m & 0 \end{pmatrix} \underline{b} \right) \in \mathbb{Z}$$

for all $\underline{a}, \underline{b} \in \mathcal{M} \subset (F \bigotimes_{\mathbb{Q}} \mathbb{R})^{2m}$. For every $Z = (Z^1, \ldots, Z^e) \in \mathfrak{H}_m^e$ define a map

$$J_Z : (F \otimes_{\mathbb{Q}} \mathbb{R})^{2m} = \mathbb{R}^{2g} \to \mathbb{C}, \ \underline{a} \mapsto \mathrm{diag}((Z^1, \mathbb{I}_m), \ldots, (Z^e, \mathbb{I}_m))\underline{a}.$$

Hence $J_Z(\mathcal{M})$ is a lattice in $\mathbb{C}^g$ and the quotient

$$X_Z := \mathbb{C}^g / J_Z(\mathcal{M})$$

is a complex torus. Define a hermitian form H_Z on $\mathbb{C}^g$ by

$$H_Z(x, y) =^t x \mathrm{diag}(\mathrm{Im} Z^1, \ldots, \mathrm{Im} Z^e)^{-1} \overline{y}.$$

By definition H_Z is a positive definite hermitian form on $\mathbb{C}^g$. Moreover H_Z defines a polarization on X_Z.

Finally we have $\rho(a) J_Z(\underline{b}) = J_Z(a \cdot \underline{b})$ for all $a \in F$ and $\underline{b} \in F^{2m}$. Since $\mathcal{M} \otimes \mathbb{Q} = F^{2m}$, this yields $\rho(na) J_Z(\mathcal{M}) \subseteq J_Z(\mathcal{M})$ for some integer $n > 0$, and we obtain

$$\rho(F) \subset \mathrm{End}^0(X_Z).$$

Set $\iota_Z = \rho$ and by an immediate matrix computation shows that

$$H_Z(\iota_Z(a)x, y) = H_Z(x, \iota_Z(a)y)$$

for all $x, y \in \mathbb{C}^g$ and $a \in F$ and the Rosati involution is the adjoint operator for H, so the Rosati involution restricts to identity on F. Combining everything we get

Proposition 2.3.1 *For every $Z \in \mathfrak{H}_m^e$ the triplet (X_Z, H_Z, ι_Z) is a polarized abelian variety with endomorphism structure (F, id_F, ι).*

The action of $\mathrm{Sp}_{2m}(\mathbb{R})$ on $\mathfrak{H}_m$ induces an action of the group

$$G := \bigoplus_{i=1}^{e} \mathrm{Sp}_{2m}(\mathbb{R})$$

on $\mathfrak{H}_m^e$: for $Z = (Z^1, \dots, Z^e) \in \mathfrak{H}_m^e$ and $M = (M^1, \dots, M^e) \in G$ define

$$M(Z) = (M^1(Z^1), \dots, M^e(Z^e))$$

with $M^\nu(Z^\nu) = (\alpha^\nu Z^\nu + \beta^\nu)(\gamma^\nu Z^\nu \delta^\nu)^{-1}$, where $M^\nu = \begin{pmatrix} \alpha^\nu & \beta^\nu \\ \gamma^\nu & \delta^\nu \end{pmatrix}$ with $(m \times m)$-matrices $\alpha^\nu, \beta^\nu, \gamma^\nu$ and δ^ν. And define the subgroup $G(\mathcal{M})$ of G by

$$G(\mathcal{M}) = \left\{ M \in G \mid \mathrm{diag}({}^tM^1, \dots, {}^tM^e)\mathcal{M} \subseteq \mathcal{M} \right\}.$$

Proposition 2.3.2 *Let Z and Z' be two points in $\mathfrak{H}_m^e$. The polarized abelian varieties $(X_{Z'}, H_{Z'}, \iota_{Z'})$ and (X_Z, H_Z, ι_Z) with endomorphism structure $(F, ', \rho)$ are isomorphic if and only if there is an $M \in G(\mathcal{M})$ such that $Z' = M(Z)$.*

The group $G(\mathcal{M})$ is discrete in G, since $\mathcal{M}$ is a lattice in some real vector space on which G acts.

Proposition 2.3.3 *Any discrete subgroup $G \subseteq Sp_{2g}(\mathbb{R})$ acts properly and discontinuously on $\mathfrak{H}_g$.*

So the group acts properly and discontinuously on $\mathfrak{H}_m^e$. Hence by the following theorem the quotient

$$\mathcal{A}(\mathcal{M}) := \mathfrak{H}_m^e / G(\mathcal{M})$$

is a normal complex analytic space. We call it the moduli space of polarized abelian varieties with endomorphism structure $(F, ', \rho)$ associated to the F-module $\mathcal{M}$.

Theorem 2.3.1 *Suppose $\mathcal{X}$ is a complex analytic space and G is a group, acting properly and discontinuously on $\mathcal{X}$. The quotient $\mathcal{X}/G$ is also a complex analytic space. Moreover, if $\mathcal{X}$ is normal, so is $\mathcal{X}/G$.*

And we can computer the dimension

$$\dim \mathcal{A}(\mathcal{M}) = \mathfrak{H}_m^e = \frac{e}{2}m(m+1).$$

Proposition 2.3.4 *Every polarized abelian variety (X, H, ι) of dimension g with endomorphism structure (F, id_F, ρ) is contained in some moduli space $\mathcal{A}(\mathcal{M})$ as above.*

2.3.2 Totally Indefinite Quaternion Multiplication

Then we consider the abelian varieties with totally indefinite quaternion multiplication. Let F be a totally indefinite quaternion skew field over a totally real number field K with $[K : \mathbb{Q}] = e$. We say that an abelian variety X admits totally indefinite quaternion multiplication by F, if there is an embedding $F \hookrightarrow \mathrm{End}^0(X)$. Proposition 2.2.1 gives

$$g = 2em$$

for some integer $m \geq 1$. The e different embeddings $F \hookrightarrow \mathbb{R}$ determine isomorphisms of $F \otimes_{\mathbb{Q}} \mathbb{R}$ with $M_2(\mathbb{R})^e$, and $(F \otimes_{\mathbb{Q}} \mathbb{R})^m$ with $M_2(\mathbb{R})^{em}$. Identifying both sides, we may assume that $'$ extends to matrix transposition on every factor $M_2(\mathbb{R})$.

Define

$$\tilde{\ } : (F \otimes_{\mathbb{Q}} \mathbb{R})^m = (M_2(\mathbb{R}))^{4em} \text{ for } \forall\ \underline{a} = {}^t(\underline{a}^1, \ldots, \underline{a}^e),$$

where

$$\underline{a}^\nu = {}^t(a_1^\nu, \ldots, a_m^\nu) \in M_2(\mathbb{R})^m$$

and

$$a_\mu^\nu = \begin{pmatrix} (a_\mu^\nu)_{11} & (a_\mu^\nu)_{12} \\ (a_\mu^\nu)_{21} & (a_\mu^\nu)_{22} \end{pmatrix} \in M_2(\mathbb{R}),$$

define

$$\underline{a} \mapsto \underline{\tilde{a}} = {}^t(\underline{\tilde{a}}^1, \ldots, \underline{\tilde{a}}^e),$$

where

$$\underline{\tilde{a}}^{\nu} = \begin{pmatrix} (\underline{a}^{\nu})_{11} \\ (\underline{a}^{\nu})_{12} \\ \vdots \\ (\underline{a}^{\nu})_{22} \end{pmatrix} \in \mathbb{C}^{d^2 m}$$

and

$$(\underline{a}^{\nu})_{jk} = \begin{pmatrix} (a_1^{\nu})_{jk} \\ (a_2^{\nu})_{jk} \\ \vdots \\ (a_m^{\nu})_{jk} \end{pmatrix} \in \mathbb{C}^{m}$$

Define a representation $\rho : F \to M_g(\mathbb{C})$ by

$$\rho(a) = \mathrm{diag}(a^1 \mathbb{I}_m, \ldots, a^e \otimes \mathbb{I}_m).$$

Obviously every representation $F \longrightarrow M_g(\mathbb{C})$ satisfying the conditions of lemma 2.3.1, is equivalent to ρ.

For a pair $(\mathcal{M}, T)$, where $\mathcal{M}$ is a free $\mathbb{Z}$-submodule of F^m of rank $2g = 4em$ and T is a nondegenerate $(m \times m)$-matrix over F with ${}^tT' = -T$ (here t means transposition of the $(m \times m)$-matrix, $'$ means transposition of the entrice on T), such that

$$\mathrm{tr}_{F/\mathbb{Q}}({}^t\underline{a} T \underline{b}') \in \mathbb{Z}$$

for all $\underline{a}, \underline{b} \in \mathcal{M} \subset F^m$. It is easy to see that such pair $(\mathcal{M}, T)$ exist.

Via the embedding $F \hookrightarrow F \otimes_{\mathbb{Q}} (\mathbb{R}) = M_2(\mathbb{R})^e$, we may consider T as an element of $M_m(M_2(\mathbb{R}))^e$ and we write T in the form $(T^1, \ldots, T^e)$ with $T^{\nu} \in M_m(M_2(\mathbb{R})) = M_{2m}(\mathbb{R})$. T is nondegenerate on F, so T^{ν} are nondegenerate, and ${}^tT' = -T$, so T^{ν} are alternating matrices.

Define $P = (\mathbb{I}_m \otimes e_1, \ldots, \mathbb{I}_m \otimes e_d) \in M_{dm}(\mathbb{C})$, where $e_1, \ldots, e_d$ denotes the standard basis of $\mathbb{C}^d$, then

$$\sim\, : M_m(M_d(\mathbb{C})) \to M_d(M_m(\mathbb{C})) \text{ by } T \mapsto \widetilde{T} = {}^tPTP$$

is an isomorphism of $\mathbb{C}$-algebras.

Applying the isomorphism $\tilde{}: M_m(M_2(\mathbb{R})) \to M_2(M_m(\mathbb{R}))$, we obtain nondegenerate alternating matrices

$$\widetilde{T}^\nu \in M_2(M_m(\mathbb{R})) = M_{2m}(\mathbb{R})$$

Hence there exist matrices $W^\nu \in GL_{2m}(\mathbb{R})$ such that

$$\widetilde{T}^\nu = {}^t W^\nu \begin{pmatrix} 0 & \mathbb{I}_m \\ -\mathbb{I}_m & 0 \end{pmatrix} W^\nu$$

for $\nu = 1, \ldots, e$.

For any $Z = (Z^1, \ldots, Z^e) \in \mathfrak{H}_m^e$, define a $\mathbb{R}$-vector space homomorphism

$$J_Z : \mathbb{R}^{4em} \to \mathbb{C}^{2em} = \mathbb{C}^g$$

by the matrix

$$\mathrm{diag}(J_Z^1, \ldots, J_Z^e), \text{ where } J_Z^\nu = \begin{pmatrix} (Z^\nu, \mathbb{I}_m)W^\nu & 0 \\ 0 & (Z^\nu, \mathbb{I}_m)W^\nu \end{pmatrix}$$

Columns of the defining matrix are linearly independent over $\mathbb{R}$, so J_Z is an isomorphism. Hence $J_Z(\mathcal{M}\tilde{})$ is a lattice in $\mathbb{C}^g$. ($\mathcal{M}\tilde{}$ means consider an element $\underline{a} \in \mathcal{M}$ as $\tilde{\underline{a}}$.) The quotient

$$X_Z := \mathbb{C}^g / J_Z(\mathcal{M}\tilde{})$$

is a complex torus. Moreover, define a hermitian form H_Z on $\mathbb{C}^g$ by

$$H_Z(x, y) = {}^t x \mathrm{diag}(H^1, \ldots, H^e)\overline{y}, \text{ where } H^\nu = \begin{pmatrix} \mathrm{Im} Z^\nu & 0 \\ 0 & \mathrm{Im} Z^\nu \end{pmatrix}^{-1}$$

By definition H_Z is a positive definite hermitian form on $\mathbb{C}^g$, and by the following lemma, H_Z defines a polarization on X_Z.

Lemma 2.3.2 *For all $\underline{a}, \underline{b} \in (F \otimes_{\mathbb{Q}} \mathbb{R})^m$*

$$Im H_Z(J_Z(\tilde{\underline{a}}), J_Z(\tilde{\underline{b}})) = tr_{F\otimes_{\mathbb{Q}}\mathbb{R}/\mathbb{R}}({}^t\underline{a} T \underline{b}')$$

Finally, we can check $\rho(F) \subset \mathrm{End}^0(X_Z)$.

Set $\iota_Z = \rho$, we have

Proposition 2.3.5 *Choose $(\mathcal{M}, T)$. For every $Z \in \mathfrak{H}_m^e$ the triplet (X_Z, H_Z, ι_Z) is a polarized abelian variety with endomorphism structure $(F, ', \rho)$.*

2.3.3 Totally Definite Quaternion Multiplication

Then we consider the abelian varieties with totally definite quaternion multiplication. Let $(F, ')$, where F is a totally definite quaternion skew field over a totally real number field K with $[K : \mathbb{Q}] = e$. We say that an abelian variety X admits totally definite quaternion multiplication by F, if there is an embedding $F \hookrightarrow \mathrm{End}^0(X)$. Proposition 2.2.1 gives

$$g = 2em$$

for some integer $m \geq 1$. The e different embeddings $K \hookrightarrow \mathbb{R}$ determine isomorphisms of $F \otimes_{\mathbb{Q}} \mathbb{R}$ with $\mathbb{H}^e$.

Consider $\mathbb{H}$ as a subalgebra of $M_2(\mathbb{C})$ via the represent

$$\mathbb{H} = \mathbb{C} \oplus j\mathbb{C} \to M_2(\mathbb{C}),\ \alpha + j\beta \mapsto \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}.$$

So we have

$$F \otimes_{\mathbb{Q}} \mathbb{R} \hookrightarrow M_2(\mathbb{C})^e$$
$$(F \otimes_{\mathbb{Q}} \mathbb{R})^m \hookrightarrow M_2(\mathbb{C})^{em}.$$

We may assume $'$ extends to $x' = {}^t\bar{x}$ on every factor $M_2(\mathbb{C})$ and we identify the elements of $F \otimes_{\mathbb{Q}} \mathbb{R}$ (respectively $(F \otimes_{\mathbb{Q}} \mathbb{R})^m$) with their image in $M_2(\mathbb{C})^e$ (respectively $M_2(\mathbb{C})^{em}$). Moreover we use the isomorphism

$$M_2(\mathbb{C})^{em} \longrightarrow M_2(\mathbb{C})^{4em} \text{ by } \underline{a} \mapsto \underline{\tilde{a}}$$

as above.

Define a representation $\rho : F \to M_g(\mathbb{C})$ by

$$\rho(a) = \operatorname{diag}(a^1 \otimes \mathbb{I}_m, \ldots, a^e \otimes \mathbb{I}_m).$$

Obviously every representation $F \to M_g(\mathbb{C})$ satisfying the conditions of lemma 2.3.1, is equivalent to ρ.

Let

$$\mathcal{H}_m := \{Z \in M_m(\mathbb{C}) \mid {}^tZ = -Z, \mathbb{I}_m - {}^t\bar{Z}Z > 0\}$$

Fix a pair $(\mathcal{M}, T)$, where $\mathcal{M}$ is a free $\mathbb{Z}$-submodule of F^m of rank $2g = 4em$ and T is a nondegenerate $(m \times m)$-matrix over F with ${}^tT' = -T$, such that

$$\operatorname{tr}_{F/\mathbb{Q}}({}^t\underline{a}T\underline{b}') \in \mathbb{Z}$$

for all $\underline{a}, \underline{b} \in \mathcal{M} \subset F^m$. It is easy to see that such pair $(\mathcal{M}, T)$ exist.

Consider $M_m(F)$ as a subspace of $F \otimes_{\mathbb{Q}} \mathbb{R} = M_m(\mathbb{H})^e \subset M_m(M_2(\mathbb{C}))^e$, then T is of the form $(T^1, \ldots, T^e)$ with nondegenerate matrices T^ν contained in the image of the map $M_m(\mathbb{H}) \to M_m(M_2(\mathbb{C}))$. ${}^tT' = -T$, so T^ν is skew-hermitian for $\nu = 1, \ldots, e$. Since any nondegenerate skew-hermitian matrix in $M_m(\mathbb{H})$ is equivalent to $i\mathbb{I}_m$ over $\mathbb{H}$ ([13]) and $i\mathbb{H}$ maps to

$$\mathbb{I}_m \otimes \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \operatorname{diag}(i, -i, \ldots, i, -i) \text{ in } M_m(M_2(\mathbb{C})),$$

so there are nonsingular matrices W^ν in the image of $M_m(\mathbb{H}) \to M_m(M_2(\mathbb{C}))$, such that

$$\widetilde{T}^\nu = {}^tW^\nu \left(\mathbb{I}_m \otimes \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \right) \overline{\overline{W^\nu}}$$

for $\nu = 1, \ldots, e$.

Applying the isomorphism $\widetilde{\ } : M_m(M_2(\mathbb{C})) \to M_2(M_m(\mathbb{C}))$ as in indefinite quaternion case, we have

$$\widetilde{T}^\nu = \begin{pmatrix} \widetilde{T}^\nu_{11} & \widetilde{T}^\nu_{12} \\ -\overline{\widetilde{T}}^\nu_{21} & \overline{\widetilde{T}}^\nu_{22} \end{pmatrix} \text{ and } \widetilde{W}^\nu = \begin{pmatrix} \widetilde{W}^\nu_{11} & \widetilde{W}^\nu_{12} \\ -\overline{\widetilde{W}}^\nu_{21} & \overline{\widetilde{W}}^\nu_{22} \end{pmatrix}$$

with $\widetilde{T}^{\nu}_{jk}, \widetilde{T}^{\nu}_{jk} \in M_m(\mathbb{C})$.

For any $Z = (Z^1, \ldots, Z^e) \in \mathcal{H}^e_m$, define a $\mathbb{R}$-vector space homomorphism

$$J_Z : \mathbb{R}^{4em} \to \mathbb{C}^{2em} = \mathbb{C}^g$$

by the matrix

$$\mathrm{diag}(J^1_Z, \ldots, J^e_Z), \text{ where } J^{\nu}_Z = \begin{pmatrix} ({}^tZ^{\nu}, \mathbb{I}_m)\widetilde{W}^{\nu} & 0 \\ 0 & ({}^tZ^{\nu}, \mathbb{I}_m)\widetilde{W}^{\nu} \end{pmatrix}$$

with respect to the standard basis.

Lemma 2.3.3 *J_Z restricted to the subspace $((F \otimes_{\mathbb{Q}} \mathbb{R})^m)\tilde{}$ of $\mathbb{C}^{4em}$ is an isomorphism of $\mathbb{R}$-vector spaces.*

Hence $J_Z(\mathcal{M}\tilde{})$ is a lattice in $\mathbb{C}^g$, and the quotient

$$X_Z := \mathbb{C}^g / J_Z(\mathcal{M}\tilde{})$$

is a complex torus. Moreover, define a hermitian form H_Z on $\mathbb{C}^g$ by

$$H_Z(x, y) = 2^t x \mathrm{diag}(H^1, \ldots, H^e)\bar{y},$$

where

$$H^{\nu} = \begin{pmatrix} (\mathbb{I}_m -{}^t \overline{Z}^{\nu} Z^{\nu})^{-1} & 0 \\ 0 & (\mathbb{I}_m - \overline{Z}^{\nu}({}^tZ^{\nu}))^{-1} \end{pmatrix}$$

By definition H_Z is a positive definite hermitian form on $\mathbb{C}^g$. and by the following lemma, H_Z defines a polarization on X_Z.

Lemma 2.3.4 *For all $\underline{a}, \underline{b} \in (F \otimes_{\mathbb{Q}} \mathbb{R})^m$*

$$ImH_Z(J_Z(\tilde{\underline{a}}), J_Z(\tilde{\underline{b}})) = tr_{F\otimes_{\mathbb{Q}}\mathbb{R}/\mathbb{R}}({}^t\underline{a}T\underline{b}')$$

and for all $\underline{a}, \underline{b} \in \mathcal{M}$, $tr_{(F\otimes_{\mathbb{Q}}\mathbb{R})/\mathbb{R}}({}^t\underline{a}T\underline{b}') = tr_{F/\mathbb{Q}}({}^t\underline{a}T\underline{b}') \in \mathbb{Z}$

Finally, we can check $\rho(F) \subset \mathrm{End}^0(X_Z)$.

Set $\iota_Z = \rho$, we have

Proposition 2.3.6 *Choose $(\mathcal{M}, T)$, for every $Z \in \mathcal{H}^e_m$ the triplet (X_Z, H_Z, ι_Z) is a polarized abelian variety with endomorphism structure $(F, \ ', \rho)$.*

2.3.4 Complex Multiplication

Then we consider the abelian varieties with complex multiplication. Let $(F, ')$, where F is a skew field of degree d^2 over its center K, K is a totally complex quadratic extension of a totally real number field K_0 with $[K_0 : \mathbb{Q}] = e_0$, $'$ is a positive involution which can be extending to complex conjugation on K. We say that an abelian variety X admits complex multiplication by F, if there is an embedding $F \hookrightarrow \mathrm{End}^0(X)$ as above. Proposition 2.2.1 gives

$$g = d^2 e_0 m$$

for some integer $m \geq 1$.

Choose an isomorphism of $K_0 \otimes_{\mathbb{Q}} \mathbb{R}$ with $\mathbb{R}^{e_o}$, and extend it to isomorphisms of $F \otimes_{\mathbb{Q}} \mathbb{R}$ with $M_d(\mathbb{C})^{e_0}$, and $(F \otimes_{\mathbb{Q}} \mathbb{R})^m$ with $M_d(\mathbb{C})^{e_0 m}$. We identify the elements of $F \otimes_{\mathbb{Q}} \mathbb{R}$ (respectively $(F \otimes_{\mathbb{Q}} \mathbb{R})^m$) with their images in $M_d(\mathbb{C})^{e_0}$ (respectively $M_d(\mathbb{C})^{e_0 m}$).

We may assume the anti-involution $'$ extends to $x' = {}^t\bar{x}$ on every factor $M_d(\mathbb{C})$.

Define an $\mathbb{R}$-linear embedding

$$\tilde{\tilde{\ }} : \mathbb{C}^{d^2 e_0 m} \to \mathbb{C}^{2d^2 e_0 m} \text{ by } \underline{\tilde{a}} = \begin{pmatrix} \underline{\tilde{a}}^1 \\ \vdots \\ \underline{\tilde{a}}^{e_0} \end{pmatrix} \mapsto \underline{\tilde{\tilde{a}}} = \begin{pmatrix} \underline{\tilde{a}}^1 \\ \bar{\underline{\tilde{a}}}^1 \\ \vdots \\ \underline{\tilde{a}}^{e_0} \\ \bar{\underline{\tilde{a}}}^{e_0} \end{pmatrix}$$

Denote $\tilde{\tilde{\ }} \circ \tilde{\ } : (F \otimes_{\mathbb{Q}} \mathbb{R})^m = M_d(\mathbb{C})^{e_0 m} \to \mathbb{C}^{d^2 e_0 m} \to \mathbb{C}^{2d^2 e_0 m}$ by $\tilde{\tilde{\ }}$ for abbreviation.

Fix integers $r_1, s_1, \ldots, r_{e_0}, s_{e_0} \geq 0$, satisfying $r_\nu + s_\nu = dm$ for $\nu = 1, \ldots, e_0$. Define a representation $\rho : F \to M_g(\mathbb{C})$ by

$$\rho(a) = \mathrm{diag}(a^1 \otimes \mathbb{I}_{s_1}, \underline{a}^1 \otimes \mathbb{I}_{r_1}, \ldots, a^{e_0} \otimes \mathbb{I}_{s_{e_0}}, \underline{a}^{e_0} \otimes \mathbb{I}_{r_{e_0}}).$$

By $r_\nu + s_\nu = dm$, we can see that ρ satisfies the lemma 2.3.1, and every representation $F \to M_g(\mathbb{C})$ satisfying lemma 2.3.1 is equivalent to this for some integers r_ν, s_ν as above.

Denote

$$\mathcal{H}_{r,s} := \{Z \in M(r \times s, \mathbb{C}) \mid \mathbb{I}_s -{}^t \overline{Z} Z > 0\}$$

In case $rs = 0$, let $\mathcal{H}_{r,s}$ be the space consisting of a simple point, which we denote by 0.

For a pair $(\mathcal{M}, T)$, where $\mathcal{M}$ is a free $\mathbb{Z}$-submodule of F^m of rank $2g = 2d^2e_0m$ and T is a nondegenerate matrix in $M_m(F)$ with ${}^tT' = -T$ and signature $((r_1, s_1), \ldots, (r_{e_0}, s_{e_0}))$ such that

$$\mathrm{tr}_{F/\mathbb{Q}}({}^t\underline{a}T\underline{b}') \in \mathbb{Z}$$

for all $\underline{a}, \underline{b} \in \mathcal{M}$. Here signature of T means: Consider

$M_m(F)$ as a subspace of $M_m(F \otimes_{\mathbb{Q}} \mathbb{R}) = M_m(M_d(\mathbb{C}))^{e_0}$, the matrices T is of the form $(T^1, \ldots, T^{e_0})$ with nondegenerate matrices $T^\nu \in M_m(M_d(\mathbb{C})) = M_{dm}(\mathbb{C})$. Moreover ${}^tT' = -T$ means T^ν is skew-hermitian as an element of $M_{dm}(\mathbb{C})$, define T with signature of $((r_1, s_1), \ldots, (r_{e_0}, s_{e_0}))$ if T^ν are of signature (r_ν, s_ν) for $\nu = 1, \ldots, e_0$. It is easy to see that such pair $(\mathcal{M}, T)$ exist.

T^ν are nondegenerate skew-hermitian with signature (r_ν, s_ν), then there exist matrices $W^\nu \in \mathrm{GL}_{dm}(\mathbb{C})$ such that

$$\widetilde{T}^\nu = {}^t W^\nu \begin{pmatrix} i\mathbb{I}_{r_\nu} & 0 \\ o & -i\mathbb{I}_{s_\nu} \end{pmatrix} \overline{W}^\nu$$

for any $\nu = 1, \ldots, e_0$.

For any $Z = (Z^1, \ldots, Z^{e_0}) \in \mathcal{H}_{r_1,s_1} \times \ldots \times \mathcal{H}_{r_{e_0},s_{e_0}}$, define a $\mathbb{C}$-vector space homomorphism

$$J_Z : \mathbb{C}^{2d^2e_0m} \to \mathbb{C}^{d^2e_0m} = \mathbb{C}^g$$

by the matrix

$$\mathrm{diag}(J_Z^1, \ldots, J_Z^{e_0}),$$

where

$$J_Z^\nu = \begin{pmatrix} \mathbb{I}_d \otimes ({}^tZ^\nu, \mathbb{I}_{s_\nu})W^\nu & 0 \\ 0 & \mathbb{I}_d \otimes (\mathbb{I}_{r_\nu}, Z^\nu)\overline{W}^\nu \end{pmatrix}$$

with respect to the standard basis.

Lemma 2.3.5 *J_Z restricts to the subspace $((F \otimes_\mathbb{Q} \mathbb{R})^m)^{\tilde{\tilde{}}}$ of $\mathbb{C}^{2d^2 e_0 m}$ is an isomorphism of $\mathbb{R}$-vector spaces.*

Hence $J_Z(\mathcal{M}^{\tilde{\tilde{}}})$ is a lattice in $\mathbb{C}^g$, and the quotient

$$X_Z := \mathbb{C}^g / J_Z(\mathcal{M}^{\tilde{\tilde{}}})$$

is a complex torus. Moreover, define a hermitian form H_Z on $\mathbb{C}^g$ by

$$H_Z(x, y) = 2\ {}^tx\ \mathrm{diag}(H^1, \ldots, H^{e_0})\bar{y},$$

where

$$H^\nu = \begin{pmatrix} \mathbb{I}_d \otimes (\mathbb{I}_{s_\nu} -{}^t\overline{Z}^\nu Z^\nu)^{-1} & 0 \\ 0 & \mathbb{I}_d \otimes (\mathbb{I}_{r_\nu} - \overline{Z}^\nu({}^tZ^\nu))^{-1} \end{pmatrix}$$

By definition H_Z is a positive definite hermitian form on $\mathbb{C}^g$, and by the following lemma, H_Z defines a polarization on X_Z.

Lemma 2.3.6 *For all $\underline{a}, \underline{b} \in (F \otimes_\mathbb{Q} \mathbb{R})^m$*

$$ImH_Z(J_Z(\tilde{\tilde{\underline{a}}}), J_Z(\tilde{\tilde{\underline{b}}})) = tr_{F\otimes_\mathbb{Q}\mathbb{R}/\mathbb{R}}({}^t\underline{a}T\underline{b}')$$

Finally we can check for all $a \in F, \bar{b} \in (F \bigotimes_\mathbb{Q} \mathbb{R})^m$, we have

$$\rho(a)J_Z(\tilde{\tilde{\underline{b}}}) = J_Z(a \cdot \tilde{\tilde{\underline{b}}})$$

Since $\mathcal{M} \otimes \mathbb{Q} = F^m$ for any $a \in F$, there is an $n > 0$ such that $\rho(na)J_Z(\tilde{\tilde{\mathcal{M}}}) \subseteq J_Z(\tilde{\tilde{\mathcal{M}}})$, so we have $\rho(F) \subset \mathrm{End}^0(X_Z)$.

Set $\iota_Z = \rho$, we have

Proposition 2.3.7 *Fix $((r_1, s_1), \ldots, (r_{e_0}, s_{e_0}))$, choose $(\mathcal{M}, T)$, for every $Z \in \mathcal{H}_{r_1,s_1} \times \ldots \times \mathcal{H}_{r_{e_0},s_{e_0}}$ the triplet (X_Z, H_Z, ι_Z) is a polarized abelian variety with endomorphism structure $(F,\ ', \rho)$.*

2.3.5 Shimura Varieties

Now we will consider the moduli of the last three cases.

Let $(F, ', \rho)$ be a type of an endomorphism structure and $(\mathcal{M}, T)$ a pair as in the last three cases respectively. Let

$$\mathcal{H} = \begin{cases} \mathfrak{H}_m \times \ldots \times \mathfrak{H}_m & \text{for the totally indefinite multiplication type} \\ \mathcal{H}_m \times \ldots \times \mathcal{H}_m & \text{for the totally definite multiplication type} \\ \mathcal{H}_{r_1,s_1} \times \ldots \times \mathcal{H}_{r_{e_0},s_{e_0}} & \text{for the complex multiplication type} \end{cases}$$

and

$$G = \begin{cases} \mathrm{Sp}_{2m}(\mathbb{R}) \times \ldots \times \mathrm{Sp}_{2m}(\mathbb{R}) & \text{for the totally indefinite multiplication type} \\ \widetilde{U}_{m,m} \times \ldots \times \widetilde{U}_{m,m} & \text{for the totally definite multiplication type} \\ U_{r_1,s_1} \times \ldots \times U_{r_{e_0},s_{e_0}} & \text{for the complex multiplication type} \end{cases}$$

Where

$$\widetilde{U}_{m,m} = \left\{ M = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \in M_{2m}(\mathbb{C}) \mid {}^tM \begin{pmatrix} \mathbb{I}_m & 0 \\ 0 & -\mathbb{I}_m \end{pmatrix} \overline{M} = \begin{pmatrix} \mathbb{I}_m & 0 \\ 0 & -\mathbb{I}_m \end{pmatrix} \right\}$$

and

$$U_{r,s} = \left\{ M_{r+s}(\mathbb{C}) \mid {}^tM \begin{pmatrix} \mathbb{I}_r & 0 \\ 0 & -\mathbb{I}_s \end{pmatrix} \overline{M} = \begin{pmatrix} \mathbb{I}_r & 0 \\ 0 & -\mathbb{I}_s \end{pmatrix} \right\}$$

The actions of $\mathrm{Sp}_{2m}(\mathbb{R})$ on $\mathfrak{H}_m$, of $\widetilde{U}_{m,m}$ on $\mathcal{H}_m$, and of U_{r_ν,s_ν} on $\mathcal{H}_{r_\nu,s_\nu}$, $\nu = 1, \ldots, e_0$, induce an action of G on $\mathcal{H}$: For $Z = (Z^1, \ldots, Z^{e_0}) \in \mathcal{H}$ and $M = (M^1, \ldots, M^{e_0}) \in G$ define

$$M(Z) = (M^1(Z^1), \ldots, M^{e_0}(Z^{e_0}))$$

with

$$M^\nu(Z^\nu) = (\alpha^\nu Z^\nu + \beta^\nu)(\gamma^\nu Z^\nu + \delta^\nu)^{-1},$$

where $M^\nu = \begin{pmatrix} \alpha^\nu & \beta^\nu \\ \gamma^\nu & \delta^\nu \end{pmatrix}$ with $(m \times m)$-matrices $\alpha^\nu, \beta^\nu, \gamma^\nu$ and δ^ν for the first tow types and an $(r_\nu \times r_\nu)$-matrix α^ν and an $(s_\nu \times s_\nu)$-matrix δ^ν for the last type.

Define

$$G(\mathcal{M}, T) = \{M = (M^1, \dots, M^{e_0}) \in G \mid$$
$$\mathrm{diag}(\mathbb{I}_d \otimes (\widetilde{W}^1\, {}^t M^1 \widetilde{W}^1)^{-1}, \dots, \mathbb{I}_d \otimes (\widetilde{W}^{e_0}\, {}^t M^{e_0} \widetilde{W}^{e_0})^{-1})\mathcal{M}^{\check{}} \subset \mathcal{M}^{\check{}}\Big\}$$

One can prove

Proposition 2.3.8 *Let Z and Z' be two points of $\mathcal{H}$. The polarized abelian varieties $(X_{Z'}, H_{Z'}, \iota_{Z'})$ and (X_Z, H_Z, ι_Z) with endomorphism structure $(F, {}', \rho)$ are isomorphic if and only if there is an $M \in G(\mathcal{M}, T)$ such that $Z' = M(Z)$.*

The normal complex spaces

$$\mathcal{A}(\mathcal{M}, T) = \mathcal{H}/G(\mathcal{M}, T)$$

are moduli spaces of polarized abelian varieties with endomorphism structure $(F, {}', \rho)$ associated to the pair $(\mathcal{M}, T)$. We call these spaces Shimura varieties.

2.4 The Endomorphism Algebra of A General Member

Let $\mathcal{A}(\mathcal{M}, T)$ be the moduli spaces of polarized abelian varieties with $(F, {}', \rho)$ associated to $(\mathcal{M}, T)$. We have the following theorem about the endomorphism algebra of a general member (X, H, ι). (Here a general member means a member of $\mathcal{A}(\mathcal{M}, T)$ outside an union of countably many proper analytic subspaces, which can be given explicitly.)

Theorem 2.4.1 *For a general member (X, H, ι) of the space $\mathcal{A}(\mathcal{M}, T)$ associated to $(F, {}', \rho)$, we have*

$$End^0(X) = \iota(F)$$

except in the following cases:

a) $(F, ', \rho)$ *is of totally definite quaternion type and* $m \leq 2$.

b) $(F, ', \rho)$ *is of complex multiplication type and* $\sum_{\nu=1}^{e_0} r_\nu s_\nu = 0$.

c) $(F, ', \rho)$ *is of complex multiplication type and* $r_\nu = s_\nu = 1$ *for* $\nu = 1, \ldots, e_0$.

For matrices $\widetilde{W}^\nu, \nu = 1, \ldots, e_0$, associated to T as above. In case of totally real multiplication let $\widetilde{W}^\nu = -\mathbb{I}_{2m}$ for all ν. For a matrix $R \in M_{2m}(F)$ in totally real multiplication case, respectively $M_m(F)$ otherwise with ${}^tR'\mathcal{M} \subset \mathcal{M}$, define

$$M^\nu = ({}^t\widetilde{W}^\nu)^{-1}\widetilde{R}^\nu({}^t\widetilde{W}^\nu) = \begin{pmatrix} \alpha^\nu & \beta^\nu \\ \gamma^\nu & \delta^\nu \end{pmatrix}$$

for all ν. Then the equation

$$Z(\gamma^\nu Z + \delta^\nu) = \alpha^\nu Z + \beta^\nu$$

defines an analytic subspace of the space $\mathfrak{H}_m$ for the first tow type, $\mathcal{H}_m$ for the third type and $\mathcal{H}_{r_\nu, s_\nu}$ for the last type respectively, which we denote by $S^{\nu(R)}$.

Proposition 2.4.1 *If* R *is not an element of the center of* $M_{2m}(F)$ *respectively* $M_m(F)$*, then for every* ν *the set* $S^\nu(R)$ *is a proper analytic subspace of* $\mathfrak{H}_m$ *or* $\mathcal{H}_{r_\nu, s_\nu}$ *respectively, except in case a), b) of theorem 2.4.1.*

Proposition 2.4.2 *If* $Z \in \mathcal{H} - S$*, then* $\iota_Z(K)$ *is the centralizer of* $\iota_Z(F)$ *in* $End^0(X_Z)$.

Proposition 2.4.3 *Let* (X, H_X, ι_X) *be a polarized abelian variety with* $(F, ', \rho)$*. If* $\iota_X(K)$ *is the centralizer of* $\iota_X(F)$ *in* $End^0(X)$*, then*

$$End^0(X) = \iota_X$$

except in the case c) of theorem 2.4.1.

So for a general member $(X, H, \iota) \in \mathcal{A}(\mathcal{M}, T) = \mathcal{H}/G(\mathcal{M}, T)$ besides a), b), c) of theorem 2.4.1, means $\exists Z \leftrightarrow (X, H, \iota)$ (Z uniquely determined by an action of $M \in G(\mathcal{M}, T)$) and $Z \in \mathcal{H} - S$, (S is an union of countably many proper analytic subspaces by proposition 2.4.1) so by proposition 2.4.2 $\iota_Z(K)$ is the centralizer of $\iota_Z(F)$ in $\mathrm{End}^0(X_Z)$, then by proposition 2.4.3 $\mathrm{End}^0(X) = \iota_X(F)$.

This is the theorem 2.4.1.

Chapter 3

Jacobians of Some Families of Curves

We will consider the families of curves arising in [4].

3.1 Some Families of Curves

Let $f : \chi \longrightarrow S$ be universal families of $N \geq 4$ points in $\mathbb{P}^1$, and let $g : \mathcal{Z} \to S$ be the families of the d-th cyclic covers of $\mathbb{P}^1$ ramified on χ. According to the classification of all possibilities of μ_s in [4], we would consider the following families of curves defined over $\mathbb{C}$.

$$\begin{aligned}
y^d &= x(x-1)(x-\lambda_1), d > 4 \\
y^d &= x(x-1)(x-\lambda_1)(x-\lambda_2), d = 3,4,5,6,8,9,10,12,18 \\
y^d &= x(x-1)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3), d = 3,4,6,8,12 \\
y^4 &= x(x-1)(x-\lambda_1)\ldots(x-\lambda_{N-3}), N = 7,8.
\end{aligned}$$

For simplicity, we will use $\mu = N-3 \geq 1$ to denote the maximal index of λ.

Let $g : \mathcal{Z} \to S$ be the families as above, the Galois group $G = \mathbb{Z}/d$ acts fibrewise by automorphisms on the families $g : \mathcal{Z} \to S$. We consider the induced families $g : \mathrm{Jac}(\mathcal{Z}/S) \to S$ of Jacobians. Let $\xi = e^{\frac{2\sqrt{-1}\,\pi}{d}}$, then $\mathbb{Z}[t]/(t^{d-1}+\ldots+1)$ acts on $\mathrm{Jac}(\mathcal{Z}/S) \to S$ via the action of ξ on $g : \mathcal{Z} \to S$.

Claim 3.1.1 *The ring $\mathbb{Z}[t]/(t^{d-1}+\ldots+1)$ acts as a subring of the endomorphism ring of $Jac(\mathcal{Z}/S) \to S$.*

Proof. We only need check at any $s \in S$, $\mathbb{Z}[t]/(t^{d-1}+\ldots+1)$ acts as a subring of the endomorphism algebra of $\mathrm{Jac}(Z_s)$.

Let $\pi : Z_s \to \mathbb{P}^1$ be the cyclic cover of degree d ramified on $\mu+3$ points, and $G = \mathbb{Z}/d$ be the Galois group acts Z_s, hence acts (by transport of structure) on $\pi_*\mathbb{C}$. Then π induces

$$\pi\mid_{Z_s-\pi^{-1}(\{0,1,\infty,\lambda_1,\ldots,\lambda_\mu\})}: Z_s\mid_{Z_s-\pi^{-1}(\{0,1,\infty,\lambda_1,\ldots,\lambda_\mu\})}\longrightarrow \mathbb{P}^1-\{0,1,\infty,\lambda_1,\ldots,\lambda_\mu\},$$

we still denote as π, which is a étale map.

Because ∞ is in the branch locus, this means that the function field $\mathbb{C}(Z_s)$ of Z_s is a subextension of the extension $\mathbb{C}(\mathbb{P}^1)((z-s)^{1/d}_{s\in(\{0,1,\infty,\lambda_1,\ldots,\lambda_\mu\})-\{\infty\}})$ of $\mathbb{C}(z) = \mathbb{C}(\mathbb{P}^1)$. At a point $z \in \{0,1,\infty,\lambda_1,\ldots,\lambda_\mu\}$, the representation of G on $(\pi_*\mathbb{C})_z = H^0(\pi^{-1}(z),\mathbb{C})$ is a regular representation of G. For each character χ of G, let L_χ be the subsheaf of $\pi_*\mathbb{C}$ on which G acts by χ. One has

$$\pi_*\mathbb{C} = \bigoplus_\chi L_\chi$$

and L_χ is a rank on local system on $\mathbb{P}^1-\{0,1,\infty,\lambda_1,\ldots,\lambda_\mu\}$.

In fact, if let $\sigma \in G$ be the generator corresponding to $\xi = e^{\frac{2\sqrt{-1}\,\pi}{d}}$, then χ can be identified with $\xi^i = e^{\frac{2i\sqrt{-1}\,\pi}{d}}$ ($\chi : G \to \mathbb{C}^*$ is uniquely determined by $\sigma \mapsto \xi^i$), so L_χ is the local system with monodromy $\{\alpha^\nu\}$, where $\alpha^\nu = e^{\frac{2i\sqrt{-1}\,\pi}{d}}$ if $\nu = 0,1,\lambda_1,\ldots,\lambda_\mu$ and $\alpha^\infty = e^{\frac{2i\sqrt{-1}\,\pi(2d-(\mu+2))}{d}}$, which we will denote as L_{ξ^i}.

So we have

$$\begin{aligned} H^1(Z_s,\mathbb{C}) &\cong H^1(\mathbb{P}^1-\{0,1,\infty,\lambda_1,\ldots,\lambda_\mu\},(\pi_s)_*\mathbb{C}) \\ &\cong \bigoplus_{i=1}^{d-1} H^1(\mathbb{P}^1-\{0,1,\infty,\lambda_1,\ldots,\lambda_\mu\},L_{\xi^i}) \end{aligned}$$

$$(\because H^1(\mathbb{P}^1 - \{0, 1, \infty, \lambda_1, \dots, \lambda_\mu\}, L_{\xi^0}) \cong H^1(\mathbb{P}^1, \mathbb{C}) = 0)$$

and σ acts on $H^1(\mathbb{P}^1 - \{0, 1, \infty, \lambda_1, \dots, \lambda_\mu\}, L_{\xi^i})$ by ξ^i.

Use the notation in [4] section 2.5. Take a partition of $\{0, 1, \infty, \lambda_1, \dots, \lambda_\mu\}$ by $\{0\} \bigcup \{1, \infty, \lambda_1, \dots, \lambda_\mu\}$, T a tree (a tree is a contractible CW complex of dimension ≤ 1), and embedding $\beta : T \hookrightarrow \mathbb{P}^1$, mapping the set of vertices of T onto $\{1, \infty, \lambda_1, \dots, \lambda_\mu\}$. Because $\mu \geq 1$, we can take an open edge $a \in T$ such that

$$\lim_{t \to 0} a(t) = 1, \lim_{t \to 1} a(t) = \lambda_1,$$

and $l_i(a) \in H^0(a, \beta^* L_{\xi^i}^\vee)$. By [4] proposition 2.5.1, $l_i(a) \cdot \beta \mid a \neq 0$ in $H_1^{lf}(L_{\xi^i}^\vee) \cong H^1(L_{\xi^i})$, so $l_i(a) \cdot \beta \mid a, (i = 1, \dots, d-1)$ is a subset of the basis of $\bigoplus_{i=1}^{d-1} H^1(L_{\xi^i})$, hence a subset of basis of $H^1(Z_s, \mathbb{C})$.

Now assume $\sigma^i, (i = 1, \dots, d-1)$ are linear dependent, then there exists $a_i, (i = 1, \dots, d-1)$ such that $\sum_{i=1}^{d-1} a_i \sigma^i = 0$, this means

$$\sum_{i=1}^{d-1} a_i \sigma^i (l_j(a) \cdot \beta \mid a) = \sum_{i=1}^{d-1} a_i \xi^{ij} (l_j(a) \cdot \beta \mid a) = 0$$

for any $j = 1, \dots, d-1$, so we have the following equations for $j = 1, \dots, d-1$

$$\sum_{i=1}^{d-1} \xi^{ij} a_i = 0.$$

But $\det((\xi^{ij})_{i,j=1,\dots,d-1}) \neq 0$, so $a_i = 0$, for all $i = 1, \dots, d-1$, this means $\sigma^i, (i = 1, \dots, d-1)$ are linear independent in the endomorphism algebra of $\mathrm{Jac}(Z_s)$, so $\mathbb{Z}[\sigma] \cong \mathbb{Z}[t]/(t^{d-1} + \dots + 1)$ is a subring of the endomorphism algebra of $\mathrm{Jac}(Z_s)$.

Remark: In the proof of this claim, we have got the eigenspaces decomposition of

$$H^1(Z_s, \mathbb{C}) \cong \bigoplus_{i=1}^{d-1} H^1(\mathbb{P}^1 - \{0, 1, \infty, \lambda_1, \dots, \lambda_\mu\}, L_{\xi^i}),$$

where $H^1(L_{\xi^i})$ is the subspace of $H^1(Z_s, \mathbb{C})$ on which σ acts by ξ^i. (Note all of the eigenvalues are $\xi^i, i = 1, \dots, d-1$.)

Corollary 3.1.1 *Let C be the curve $y^d = x^n - 1$, if $(d,n) = 1$ and $d, n \geq 3$, then*

$$\mathbb{Q}[t]/(t^{d-1} + \ldots + 1) \otimes_{\mathbb{Q}} \mathbb{Q}[t]/(t^{n-1} + \ldots + 1) \hookrightarrow End^0(Jac(C))$$

Proof. By the above claim we know $\mathbb{Q}[t]/(t^{d-1} + \ldots + 1) \hookrightarrow \mathrm{End}^0(\mathrm{Jac}(C))$ and $\mathbb{Q}[t]/(t^{n-1} + \ldots + 1) \hookrightarrow \mathrm{End}^0(\mathrm{Jac}(C))$. Let σ and τ denote the generators of $\mathbb{Q}[\sigma] \cong \mathbb{Q}[t]/(t^{d-1} + \ldots + 1)$ and $\mathbb{Q}[\tau] \cong \mathbb{Q}[t]/(t^{n-1} + \ldots + 1)$ respectively, it is easy to see $\sigma\tau = \tau\sigma$, we only need show that $\{\sigma^i\tau^j \mid i = 1, \ldots, d-1, j = 1, \ldots, n-1\}$ are linear independent in $\mathrm{End}^0(\mathrm{Jac}(C))$. In fact $H^1(C, \mathbb{C}) \cong \mathbb{C}^{(d-1)(n-1)}$ because $(d,n) = 1$, and we can prove that the isomorphism $(x,y) \mapsto (\xi x, \zeta y)$ has a decomposition

$$H^1(C,\mathbb{C}) \cong \bigoplus_{i,j} \left(H^1(\mathbb{P}^1 - \{0,1,\infty,\lambda_1,\ldots,\lambda_\mu\}, L_{\xi^i}) \right.$$
$$\left. \cap H^1(\mathbb{P}^1 - \{0,1,\infty,\lambda_1,\ldots,\lambda_\mu\}, L_{\zeta^j})\right),$$

where ξ and ζ are the primitive roots of $t^{d-1} - 1 = 0$ and $t^{n-1} - 1 = 0$ respectively, and $H^1(L_{\xi^i}) \cap H^1(L_{\zeta^j}) \cong \mathbb{C}$. Let $e_{i,j} \in H^1(C, \mathbb{C})$ be the basis of $H^1(L_{\xi^i}) \cap H^1(L_{\zeta^j})$, as before let $\Sigma_{i,j} a_{i,j} \sigma^i \tau^j = 0$, from $\Sigma_{i,j} a_{i,j} \sigma^i \tau^j e_{k,l} = 0$, we have

$$\Sigma_{i,j} a_{i,j} \xi^{ik} \zeta^{jl} e_{k,l} = 0,$$

then we have the equations

$$\Sigma_{i,j} a_{i,j} \xi^{ik} \zeta^{jl} = 0,$$

but

$$\det(\xi^{ik}\zeta^{jl})_{(i,j)\times(k,l)} = \det(\xi^{ik})_{i\times j} \times \det(\zeta^{jl})_{j\times l} \neq 0,$$

so $a_{i,j} = 0$ for all $i = 1, \ldots, d-1, j = 1, \ldots, n-1$. This means the result.

Corollary 3.1.2 *Let σ act on the curve $C : y^d = x(x^n - 1)$ (where $n \geq 1$ and $(d, n+1) = 1$) by $\sigma(x,y) = (\xi^d x, \xi y)$, where ξ is a primitive root of $t^{dn} - 1 = 0$. Then*

$$\mathbb{Q}[t]/(t^{n(d-1)} + t^{n(d-2)} + \ldots + 1) \hookrightarrow End^0(Jac(C)).$$

Proof. As in the above corollary σ acts on C has a decomposition as

$$H^1(C,\mathbb{C}) \cong \bigoplus_{i=1,\dots,d-1,j=1,\dots,n} H^1(\mathbb{P}^1 - \{0,1,\infty,\lambda_1,\dots,\lambda_\mu\}, L_{i,j}),$$

where $L_{i,j}$ is a local system of dimension 1 with σ acts on it by $\xi^{i+(j-1)d}$. Just as the above, we have

$$\mathbb{Q}[t]/(t^{n(d-1)} + t^{n(d-2)} + \ldots + 1) \hookrightarrow \mathrm{End}^0(\mathrm{Jac}C).$$

The intersection form $<,>$ on the $\mathbb{Q}$-variation of Hodge structures $R^1 g_* \mathbb{Q}_{\mathrm{Jac}(\mathcal{Z})}$ is defined by taking cup product of 1-formes along the fibres of $g : \mathcal{Z} \to S$.

Claim 3.1.2 ([16]) *For* $l \in \mathbb{Z}[t]/(t^{d-1} + \ldots + 1)$ *and for all* $u, v \in R^1 g_* \mathbb{Q}_{Jac(\mathcal{Z})} |_{S_0} = H^1(g^{-1}(s_0), \mathbb{Q})$ *one has* $< lu, v > = < u, \bar{l}v >$.

Proof. Let σ be a generator of $G = \mathbb{Z}/n$. Then

$$< \sigma u, \sigma v > = < u, v >, \forall u, v \in H^1(g^{-1}(s_0), \mathbb{Q})$$

Let

$$H^1(g^{-1}(s_0), \mathbb{Q}) = \bigoplus_{i=1}^{n-1} V_i,$$

the decomposition in eigenspaces, i.e. $\sigma(v) = \xi^i v$ for all $v \in V_i$. Then $< V_i, V_j > = 0$ for all i, j with $i + j \neq n$. On the other hand, for $u \in V_i$ and $v \in V_{n-i}$, the equality $\overline{\sigma} = \sigma^{-1}$, implies that

$$< \sigma u, v > = \xi^i < u, v > = < u, \xi^i v > = < u, (\xi^{n-i})^{-1} v > = < u, \sigma^{-1} v >$$
$$= < u, \overline{\sigma} v > .$$

Now we will use the notations of [3].

Fix a $\lambda_0 = (\lambda_{1,0}, \dots, \lambda_{\mu,0}) \in S$ and consider the corresponding fibre of the family:

$$C_{\lambda_0} : y^d = x(x-1)(x-\lambda_{1,0})\dots(x-\lambda_{\mu,0}).$$

We have the ring $\mathbb{Z}[\sigma] = \mathbb{Z}[t]/(t^{d-1} + \ldots + 1)$ acts on $\mathrm{Jac}(C_{\lambda_0})$. Let $L = \mathbb{Q}[\sigma] \cong \mathbb{Z}[t]/(t^{d-1} + \ldots + 1) \otimes \mathbb{Q}$, then L is a semi-simple commutative algebra over $\mathbb{Q}$, so L is the direct sum of fields over $\mathbb{Q}$ by the construction theorem of simple algebra over $\mathbb{Q}$. Now L acts on $H^1(C_{\lambda_0}, \mathbb{Q})$ via the automorphism of C_{λ_0} and $< lu, v > = < u, \bar{l}v >$ for all $l \in L$ and all $u, v \in H^1(C_{\lambda_0}, \mathbb{Q})$. We are therefore in the situation of [3] 4.9.

Let $G \subset GL_L(H^1(C_{\lambda_0}, \mathbb{Q}))$ be the algebraic group over $\mathbb{Q}$ defined by Deligne. The group $G(\mathbb{Q})$ is the set of $g \in GL_L(H^1(C_{\lambda_0}, \mathbb{Q}))$ such that there is a $\mu(g) \in \mathbb{Q}^*$ satisfying:

$$< gu,\ gv > = \mu(g) < u,\ v > \ \forall u, v \in H^1(C_{\lambda_0}, \mathbb{Q}).$$

Because $H^1(C_{\lambda_0}, \mathbb{Q})$ is the cohomology of an algebraic curve, it is equipped with a Hodge structure for which $<,>$ is a polarization. The elements of L act as endomorphisms of this Hodge structure. It follows that the map

$$h_0 : S \to GL_L(H^1(C_{\lambda_0}, \mathbb{Q}))_{\mathbb{R}}$$

defining this Hodge structure factors through $G_{\mathbb{R}}$. Therefore we have a map

$$h_0 : S \to G_{\mathbb{R}}.$$

As in [5], for some proper compact open subgroup $K \subset G(\mathbf{A}_f)$ the quotient ${}_K M_{\mathbb{C}}(G, h_0)$ is the moduli space of isomorphism classes of principally polarized abelian varieties, together with the given $\mathbb{Z}[t]/(t^{d-1} + \ldots + 1)$-action satisfying the property in the above claim and a level 1 structure.

3.2 Kodaira-Spencer Map

Let $g : \mathcal{Z} \longrightarrow S$ be the families stated at the beginning of this section, the Kodaira-Spencer map is

$$\mathrm{Kod} : \Theta_S \longrightarrow R^1 g_* \Theta_{\mathcal{Z}/S}.$$

Claim 3.2.1 *The Kodaira-Spencer map of all of the families $g : \mathcal{Z} \longrightarrow S$ stated at the beginning of this chapter are injective besides the case for $d = 4$ and $\mu = 2$. (in fact, this degenerates to the case $d = 4$ and $\mu = 1$)*

Proof. The case for $\mu = 1$ and $3 \nmid d$.

Let $g : \mathcal{Z} \longrightarrow S$ be the universal family of the d-th cyclic covers of $\mathbb{P}^1$ ramified on four points in $\mathbb{P}^1$, this is just the family of

$$y^d = x(x-1)(x-\lambda_1), d > 4, 3 \nmid d$$

Given a vector field $\frac{\partial}{\partial\lambda_1}$ of Θ_S on some open set of S, in order to compute the $\mathrm{Kod}(\frac{\partial}{\partial\lambda_1})$, we take the covers of $Z_{\lambda_1} = g^{-1}(\lambda_1)$ by

$$U_1 \cup U_2 = C_{\lambda_1},$$

where $U_1 = C_{\lambda_1} \backslash \pi^{-1}\{0, 1, \lambda_1\}, U_2 = C_{\lambda_1} \backslash \pi^{-1}\{3x^2 - 2(\lambda_1 + 1)x + \lambda_1 = 0\}$.

And take the coordinates y and x on U_1 and U_2 respectively. Then by definition

$$\begin{aligned}\mathrm{Kod}(\frac{\partial}{\partial\lambda_1}) = (\frac{\partial}{\partial\lambda_1})_x - (\frac{\partial}{\partial\lambda_1})_y = \frac{\partial y}{\partial\lambda_1} \cdot \frac{\partial}{\partial y} + (\frac{\partial\lambda_1}{\partial\lambda_1}) \cdot (\frac{\partial}{\partial\lambda_1})_y - (\frac{\partial}{\partial\lambda_1})_y \\ = \frac{x - x^2}{dy^{d-1}} \frac{\partial}{\partial y}.\end{aligned}$$

On the other hand

$$H^1(C_{\lambda_1}, \Theta_{C_{\lambda_1}}) \cong H^0(U_1 \cap U_2, \Theta_{C_{\lambda_1}}) / \{v_1 - v_2 \mid v_i \in H^0(U_i, \Theta_{C_{\lambda_1}})\},$$

and $H^0(U_1 \cap U_2, \Theta_{C_{\lambda_1}})$ is generated by $x^m y^n (3x^2 - 2(\lambda_1 + 1)x + \lambda_1)^{-l} \frac{\partial}{\partial y}$.

Now we will consider the restrictions on the generators of $H^1(C_{\lambda_1}, \Theta_{C_{\lambda_1}})$.

First we have the following divisors

$$\begin{aligned}
\operatorname{div}(x) &= dp_0 - dp_\infty, \\
\operatorname{div}(y) &= (p_0 + p_1 + p_{\lambda_1}) - 3p_\infty, \\
\operatorname{div}(dx) &= (d-1)(p_0 + p_1 + p_{\lambda_1}) - (d+1)p_\infty, \\
\operatorname{div}(dy) &= \pi^{-1}(\operatorname{Zero}(3x^2 - 2(\lambda_1+1)x + \lambda_1)) - 4p_\infty, \\
\operatorname{div}(\frac{\partial}{\partial x}) &= -(d-1)(p_0 + p_1 + p_{\lambda_1}) + (d+1)p_\infty, \\
\operatorname{div}(\frac{\partial}{\partial y}) &= -\pi^{-1}(\operatorname{Zero}(3x^2 - 2(\lambda_1+1)x + \lambda_1)) + 4p_\infty,
\end{aligned}$$

where $p_0, p_1, p_{\lambda_1}, p_\infty$ are the ramified points of the cyclic covering. Then

$$x^m y^n (3x^2 - 2(\lambda_1+1)x + \lambda_1)^{-l} \frac{\partial}{\partial y} \in H^0(U_1 \cap U_2, \Theta_{C_{\lambda_1}})$$

$$\Longleftrightarrow md + 3n - ld \le 4.$$

For $x^m y^n (3x^2 - 2(\lambda_1+1)x + \lambda_1)^{-l} \frac{\partial}{\partial y} \in H^0(U_1 \cap U_2, \Theta_{C_{\lambda_1}})$, furthermore we have

$$\begin{aligned}
x^m y^n (3x^2 - 2(\lambda_1+1)x + \lambda_1)^{-l} \frac{\partial}{\partial y} &\in H^0(U_1, \Theta_{C_{\lambda_1}}) \\
&\Longleftrightarrow -l \ge 1 \\
x^m y^n (3x^2 - 2(\lambda_1+1)x + \lambda_1)^{-l} \frac{\partial}{\partial y} &\in H^0(U_2, \Theta_{C_{\lambda_1}}) \\
&\Longleftrightarrow md + n \ge 0 \text{ and } n \ge 0.
\end{aligned}$$

In order to compute $\operatorname{Kod}(\frac{\partial}{\partial \lambda_1})$, we may assume $l = 0$, then all of the restrictions on $x^m y^n \frac{\partial}{\partial y}$ are

$$\begin{cases}
md + 3n \le 4 & x^m y^n \frac{\partial}{\partial y} \in H^0(U_1 \cap U_2, \Theta_{C_{\lambda_1}}) \\
n < 0 & \text{otherwise } x^m y^n \frac{\partial}{\partial y} \in H^0(U_2, \Theta_{C_{\lambda_1}}) \\
\frac{3x^2 - 2(\lambda_1+1)x + \lambda_1}{y^{d-1}} \frac{\partial}{\partial y} \in H^0(U_1, \Theta_{C_{\lambda_1}}) & \text{because } -l \ge 1 \\
\frac{x^3 - (\lambda_1+1)x^2 + \lambda_1 x}{y^{d-1}} \frac{\partial}{\partial y} = y\frac{\partial}{\partial y} \in H^0(U_2, \Theta_{C_{\lambda_1}}) & \text{because } md + n \ge 0 \text{ and } n \ge 0
\end{cases}$$

Here $n = 1 - d$, so $0 \le m \le 3$ by $md - 3d \le 4$ (we may assume $m \ge 0$ after using the equation of the curve), and

$$\frac{3x^2 - 2(\lambda_1+1)x + \lambda_1}{y^{d-1}} \frac{\partial}{\partial y} = \frac{x^3 - (\lambda_1+1)x^2 + \lambda_1 x}{y^{d-1}} \frac{\partial}{\partial y} = 0$$

in $H^0(U_1 \cap U_2, \Theta_{C_{\lambda_1}})$. Because $m \leq 3$, we have

$$\frac{x}{y^{d-1}}\frac{\partial}{\partial y} = \frac{\lambda_1(\lambda_1+1)}{(2\lambda_1^2 - 2\lambda_1 + 2)y^{d-1}}\frac{\partial}{\partial y} \neq 0$$

and then

$$\mathrm{Kod}(\frac{\partial}{\partial \lambda_1}) = \frac{x - x^2}{dy^{d-1}}\frac{\partial}{\partial y} = \frac{-\lambda_1(\lambda_1 - 1)}{2d(\lambda_1^2 - \lambda_1 + 1))y^{d-1}}\frac{\partial}{\partial y} \neq 0.$$

In fact the set

$$\left\{\frac{1}{y}\frac{\partial}{\partial y}, \frac{x}{y}\frac{\partial}{\partial y}, \ldots, \frac{1}{y^{d-1}}\frac{\partial}{\partial y}, \ldots\right\}$$

is a subset of the basis of $H^1(C_{\lambda_1}, \Theta_{C_{\lambda_1}})$.

The case for $\mu = 1$ and $3 \mid d$. Let $d = 3k$ for $k \geq 2$.

In this case we have the following divisors

$$\begin{aligned}
\mathrm{div}(x) &= dp_0 - k(p_\infty + p'_\infty + p''_\infty), \\
\mathrm{div}(y) &= (p_0 + p_1 + p_{\lambda_1}) - (p_\infty + p'_\infty + p''_\infty), \\
\mathrm{div}(dx) &= (d-1)(p_0 + p_1 + p_{\lambda_1}) - (k+1)(p_\infty + p'_\infty + p''_\infty), \\
\mathrm{div}(dy) &= \pi^{-1}(\mathrm{Zero}(3x^2 - 2(\lambda_1+1)x + \lambda_1)) - 2(p_\infty + p'_\infty + p''_\infty), \\
\mathrm{div}(\frac{\partial}{\partial x}) &= -(d-1)(p_0 + p_1 + p_{\lambda_1}) + (k+1)(p_\infty + p'_\infty + p''_\infty), \\
\mathrm{div}(\frac{\partial}{\partial y}) &= -\pi^{-1}(\mathrm{Zero}(3x^2 - 2(\lambda_1+1)x + \lambda_1)) + 2(p_\infty + p'_\infty + p''_\infty),
\end{aligned}$$

where p_0, p_1, p_{λ_1}, are the totally ramified points of the cyclic covering and $p_\infty, p'_\infty, p''_\infty$ are the inverse image of ∞ .

In order to compute $\mathrm{Kod}(\frac{\partial}{\partial \lambda_1})$, we may assume $l = 0$ as above, then all of the restrictions on $x^m y^n \frac{\partial}{\partial y}$ are

$$\begin{cases}
mk + n \leq 2 & x^m y^n \frac{\partial}{\partial y} \in H^0(U_1 \cap U_2, \Theta_{C_{\lambda_1}}) \\
n < 0 & \text{otherwise } x^m y^n \frac{\partial}{\partial y} \in H^0(U_2, \Theta_{C_{\lambda_1}}) \\
\frac{3x^2 - 2(\lambda_1+1)x + \lambda_1}{y^{d-1}}\frac{\partial}{\partial y} \in H^0(U_1, \Theta_{C_{\lambda_1}}) & \text{because } -l \geq 1 \\
\frac{x^3 - (\lambda_1+1)x^2 + \lambda_1 x}{y^{d-1}}\frac{\partial}{\partial y} = y\frac{\partial}{\partial y} \in H^0(U_2, \Theta_{C_{\lambda_1}}) & \text{because } md + n \geq 0 \text{ and } n \geq 0
\end{cases}$$

Here $n = 1 - d$, so $0 \leq m \leq 3$ by $mk + n \leq 2$ because $k \geq 2$, and

$$\frac{3x^2 - 2(\lambda_1 + 1)x + \lambda_1)}{y^{d-1}} \frac{\partial}{\partial y} = \frac{x^3 - (\lambda_1 + 1)x^2 + \lambda_1 x}{y^{d-1}} \frac{\partial}{\partial y} = 0$$

in $H^0(U_1 \cap U_2, \Theta_{C_{\lambda_1}})$. Because $m \leq 3$, we have

$$\frac{x}{y^{d-1}} \frac{\partial}{\partial y} = \frac{\lambda_1(\lambda_1 + 1)}{(2\lambda_1^2 - 2\lambda_1 + 2)y^{d-1}} \frac{\partial}{\partial y} \neq 0$$

and then

$$\mathrm{Kod}(\frac{\partial}{\partial \lambda_1}) = \frac{x - x^2}{dy^{d-1}} \frac{\partial}{\partial y} = \frac{-\lambda_1(\lambda_1 - 1)}{2d(\lambda_1^2 - \lambda_1 + 1))y^{d-1}} \frac{\partial}{\partial y} \neq 0.$$

The case for $\mu = 2$ and $2 \nmid d$, $d \geq 3$.

Let $g : \mathcal{Z} \longrightarrow S$ be the universal family of the d-th cyclic covers of $\mathbb{P}^1$ ramified on five points in $\mathbb{P}^1$, this is just the family of

$$y^d = x(x-1)(x-\lambda_1)(x-\lambda_2), d \geq 3, 2 \nmid d.$$

As before we have the following divisors

$$\begin{aligned}
\mathrm{div}(x) &= dp_0 - dp_\infty, \\
\mathrm{div}(y) &= (p_0 + p_1 + p_{\lambda_1} + p_{\lambda_2}) - 4p_\infty, \\
\mathrm{div}(dx) &= (d-1)(p_0 + p_1 + p_{\lambda_1} + p_{\lambda_2}) - (d+1)p_\infty, \\
\mathrm{div}(dy) &= \pi^{-1}(\mathrm{Zero}(4x^3 - 3(\lambda_1 + \lambda_2 + 1)x^2 + 2(\lambda_1\lambda_2 + \lambda_1 + \lambda_2)x \\
&\quad -\lambda_1\lambda_2)) - 5p_\infty, \\
\mathrm{div}(\frac{\partial}{\partial x}) &= -(d-1)(p_0 + p_1 + p_{\lambda_1} + p_{\lambda_2}) + (d+1)p_\infty, \\
\mathrm{div}(\frac{\partial}{\partial y}) &= -\pi^{-1}(\mathrm{Zero}(4x^3 - 3(\lambda_1 + \lambda_2 + 1)x^2 + 2(\lambda_1\lambda_2 + \lambda_1 + \lambda_2)x \\
&\quad -\lambda_1\lambda_2)) + 5p_\infty,
\end{aligned}$$

where $p_0, p_1, p_{\lambda_1}, p_{\lambda_2}, p_\infty$ are the ramified points of the cyclic covering.

In order to compute $\mathrm{Kod}(\frac{\partial}{\partial \lambda_1})$ $\mathrm{Kod}(\frac{\partial}{\partial \lambda_2})$, for $x^m y^n \frac{\partial}{\partial y}$, we have as above

$$\begin{cases} md + 4n \leq 5 \\ n < 0. \end{cases}$$

Take $n = 1 - d$, so $0 \le m \le 4$ by $md + 4n \le 5$ because $d \ge 3$, then

$$\begin{aligned} & \frac{4x^3 - 3(\lambda_1 + \lambda_2 + 1)x^2 + 2(\lambda_1\lambda_2 + \lambda_1 + \lambda_2)x - \lambda_1\lambda_2}{y^{d-1}}\frac{\partial}{\partial y} \\ = & \frac{x^4 - (\lambda_1 + \lambda_2 + 1)x^3 + (\lambda_1\lambda_2 + \lambda_1 + \lambda_2)x^2 - \lambda_1\lambda_2 x}{y^{d-1}}\frac{\partial}{\partial y} = 0 \end{aligned}$$

in $H^0(U_1 \cap U_2, \Theta_{C_{\lambda_1,\lambda_2}})$. Because $m \le 4$, we have

$$\frac{x^2}{y^{d-1}}\frac{\partial}{\partial y} = \frac{A(\lambda_1, \lambda_2)x + B(\lambda_1, \lambda_2)}{C(\lambda_1, \lambda_2)y^{d-1}}\frac{\partial}{\partial y},$$

where A, B, C are polynomials of λ_1, λ_2 satisfying $A + B \neq C$ (obvious by computation) and $\frac{x}{y^{d-1}}\frac{\partial}{\partial y}, \frac{1}{y^{d-1}}\frac{\partial}{\partial y}$ are linear independent. Then

$$\begin{aligned} \mathrm{Kod}(\frac{\partial}{\partial\lambda_1}) &= -\frac{x(x-1)(x-\lambda_2)}{dy^{d-1}}\frac{\partial}{\partial y} = \frac{\lambda_2 x(x-1)}{dy^{d-1}}\frac{\partial}{\partial y} - \frac{x^2(x-1)}{dy^{d-1}}\frac{\partial}{\partial y}, \\ \mathrm{Kod}(\frac{\partial}{\partial\lambda_2}) &= -\frac{x(x-1)(x-\lambda_1)}{dy^{d-1}}\frac{\partial}{\partial y} = \frac{\lambda_1 x(x-1)}{dy^{d-1}}\frac{\partial}{\partial y} - \frac{x^2(x-1)}{dy^{d-1}}\frac{\partial}{\partial y}. \end{aligned}$$

So the space spanned by $\left\{\mathrm{Kod}(\frac{\partial}{\partial\lambda_1}), \mathrm{Kod}(\frac{\partial}{\partial\lambda_2})\right\}$ is the space

$$\begin{aligned} & \mathbb{C} < \frac{x(x-1)}{dy^{d-1}}\frac{\partial}{\partial y}, \frac{x^2(x-1)}{dy^{d-1}}\frac{\partial}{\partial y} > \\ = & \mathbb{C} < \frac{x(x-1)}{dy^{d-1}}\frac{\partial}{\partial y}, \frac{x-1}{dy^{d-1}}\frac{\partial}{\partial y} > \\ = & \mathbb{C} < \frac{(C-A)x + B}{dy^{d-1}}\frac{\partial}{\partial y}, \frac{x-1}{dy^{d-1}}\frac{\partial}{\partial y} > \end{aligned}$$

Because $A+B \neq C$, the dimension of this space is two, so the Kodaira-Spencer map is injective.

The case for $\mu = 2$ and $2 \mid d$, $d \ge 3$. This time we only need consider the case $d = 8$, because other cases factorize through coverings of odd degree. As before we have the following divisors

$$\begin{aligned}
\mathrm{div}(x) &= 8p_0 - 2(p_\infty + p'_\infty + p''_\infty + p'''_\infty),\\
\mathrm{div}(y) &= (p_0 + p_1 + p_{\lambda_1} + p_{\lambda_2}) - (p_\infty + p'_\infty + p''_\infty + p'''_\infty),\\
\mathrm{div}(dx) &= 7(p_0 + p_1 + p_{\lambda_1} + p_{\lambda_2}) - 3(p_\infty + p'_\infty + p''_\infty + p'''_\infty),\\
\mathrm{div}(dy) &= \pi^{-1}(\mathrm{Zero}(4x^3 - 3(\lambda_1 + \lambda_2 + 1)x^2 + 2(\lambda_1\lambda_2 + \lambda_1 + \lambda_2)x\\
&\quad -\lambda_1\lambda_2)) - 2(p_\infty + p'_\infty + p''_\infty + p'''_\infty),\\
\mathrm{div}(\frac{\partial}{\partial x}) &= -7(p_0 + p_1 + p_{\lambda_1} + p_{\lambda_2}) + 3(p_\infty + p'_\infty + p''_\infty + p'''_\infty),\\
\mathrm{div}(\frac{\partial}{\partial y}) &= -\pi^{-1}(\mathrm{Zero}(4x^3 - 3(\lambda_1 + \lambda_2 + 1)x^2 + 2(\lambda_1\lambda_2 + \lambda_1 + \lambda_2)x\\
&\quad -\lambda_1\lambda_2)) + 2(p_\infty + p'_\infty + p''_\infty + p'''_\infty),
\end{aligned}$$

where $p_0, p_1, p_{\lambda_1}, p_{\lambda_2}$ are the totally ramified points and $p_\infty, p'_\infty, p''_\infty, p'''_\infty$ are the inverse image of ∞.

In order to compute $\mathrm{Kod}(\frac{\partial}{\partial \lambda_1})$ and $\mathrm{Kod}(\frac{\partial}{\partial \lambda_2})$, for $x^m y^n \frac{\partial}{\partial y}$, we have as above

$$\begin{cases} 2m + n \leq 2 \\ n < 0. \end{cases}$$

Take $n = -7$, so $0 \leq m \leq 4$ by $2m + n \leq 2$, then

$$\begin{aligned}
&\frac{4x^3 - 3(\lambda_1 + \lambda_2 + 1)x^2 + 2(\lambda_1\lambda_2 + \lambda_1 + \lambda_2)x - \lambda_1\lambda_2}{y^7}\frac{\partial}{\partial y}\\
&= \frac{x^4 - (\lambda_1 + \lambda_2 + 1)x^3 + (\lambda_1\lambda_2 + \lambda_1 + \lambda_2)x^2 - \lambda_1\lambda_2 x}{y^7}\frac{\partial}{\partial y} = 0
\end{aligned}$$

in $H^0(U_1 \cap U_2, \Theta_{C_{\lambda_1,\lambda_2}})$. Because $m \leq 4$, we have $\frac{x}{y^{d-1}}\frac{\partial}{\partial y}, \frac{1}{y^{d-1}}\frac{\partial}{\partial y}$ are linear independent. By the method as above the dimension of the space spanned by $\left\{\mathrm{Kod}(\frac{\partial}{\partial \lambda_1}), \mathrm{Kod}(\frac{\partial}{\partial \lambda_2})\right\}$ is tow, so the Kodaira-Spencer map is injective.

The case for $\mu = 3$ and $5 \nmid d$, $d \geq 3$.

Let $g : \mathcal{Z} \longrightarrow S$ be the universal family of the d-th cyclic covers of $\mathbb{P}^1$ ramified on six points in $\mathbb{P}^1$, this is just the family of

$$y^d = x(x-1)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3), d \geq 3, 5 \nmid d$$

As before we have the following divisors

$$
\begin{aligned}
\mathrm{div}(x) &= dp_0 - dp_\infty, \\
\mathrm{div}(y) &= (p_0 + p_1 + p_{\lambda_1} + p_{\lambda_2} + p_{\lambda_3}) - 5p_\infty, \\
\mathrm{div}(dx) &= (d-1)(p_0 + p_1 + p_{\lambda_1} + p_{\lambda_2} + p_{\lambda_3}) - (d+1)p_\infty, \\
\mathrm{div}(dy) &= \pi^{-1}(\mathrm{Zero}(5x^4 - 4(\lambda_1+\lambda_2+\lambda_3+1)x^3 + 3(\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3 \\
&\quad +\lambda_1+\lambda_2+\lambda_3+1)x^2 - 2(\lambda_1\lambda_2\lambda_3+\lambda_1\lambda_2+\lambda_1\lambda_3+\lambda_2\lambda_3)x \\
&\quad +\lambda_1\lambda_2\lambda_3)) - 6p_\infty, \\
\mathrm{div}(\frac{\partial}{\partial x}) &= -(d-1)(p_0 + p_1 + p_{\lambda_1} + p_{\lambda_2} + p_{\lambda_3}) + (d+1)p_\infty, \\
\mathrm{div}(\frac{\partial}{\partial y}) &= -\pi^{-1}(\mathrm{Zero}(5x^4 - \ldots + \lambda_1\lambda_2\lambda_3)) + 6p_\infty,
\end{aligned}
$$

where $p_0, p_1, p_{\lambda_1}, p_{\lambda_2}, p_{\lambda_3}, p_\infty$ are the ramified points of the cyclic covering.

In order to compute $\mathrm{Kod}(\frac{\partial}{\partial\lambda_1})$, $\mathrm{Kod}(\frac{\partial}{\partial\lambda_2})$ and $\mathrm{Kod}(\frac{\partial}{\partial\lambda_3})$ for $x^m y^n \frac{\partial}{\partial y}$, we have as above

$$
\begin{cases} md + 5n \leq 6 \\ n < 0. \end{cases}
$$

Take $n = 1-d$, so $0 \leq m \leq 5$, then as before $\frac{x^2}{y^{d-1}}\frac{\partial}{\partial y}, \frac{x}{y^{d-1}}\frac{\partial}{\partial y}, \frac{1}{y^{d-1}}\frac{\partial}{\partial y}$ are linear independent. Then by the similar discuss we can prove the dimension of the space spanned by $\left\{\mathrm{Kod}(\frac{\partial}{\partial\lambda_1}), \mathrm{Kod}(\frac{\partial}{\partial\lambda_2}), \mathrm{Kod}(\frac{\partial}{\partial\lambda_3})\right\}$ is three, so the Kodaira-Spencer map is injective.

For the case $\mu = 7, 8$, we only need consider the case for $d = 2$ because $d = 4$ factorizes through $d = 2$, by the similar discuss we can prove that the Kodaira-Spencer map is injective too.

Remark: If $d = 3$ and $k = 1$, the family of curves are

$$
y^3 = x(x-1)(x-\lambda_1).
$$

For $\frac{x^m}{y^2}\frac{\partial}{\partial y}$, we have $km - 2 \leq 2$, so $m \leq 4$. So from

$$
\begin{cases} \frac{3x^2 - 2(\lambda_1+1)x + \lambda_1}{y^2}\frac{\partial}{\partial y} = 0, & \text{in } H^0(U_1, \Theta_{C_{\lambda_1}}) \\ \frac{x^3 - (\lambda_1+1)x^2 + \lambda_1 x}{y^2}\frac{\partial}{\partial y} = y\frac{\partial}{\partial y} = 0, \end{cases}
$$

we can deduce

$$\frac{1}{y^2}\frac{\partial}{\partial y} = 0.$$

So

$$\mathrm{Kod}(\frac{\partial}{\partial \lambda_1}) = 0.$$

This means that the family

$$y^3 = x(x-1)(x-\lambda_1).$$

is rigid. This can be seen from the following three ways.

a). $g = 1$ and $\mathbb{Q}[\xi]$, where ξ is a primitive root of $t^2+t+1=0$, is a totally complex number field. By claim 3.1.1 $\mathbb{Q}[\xi] \hookrightarrow \mathrm{End}^0(\mathrm{Jac}(C_{\lambda_1}))$, so the Jacobian of C_{λ_1} is of complex multiplication type, so the family is rigid.

b). Take a transformation on the x-axis such that

$$0 \mapsto 0, 1 \mapsto 1, \lambda_1 \mapsto \infty$$

and then the family is transformed to $y^3 = x(x-1)$. This is obviously rigid.

c). Later we will see this is a rigid family by looking at the eigenspaces decomposition

$$H^1(Z_s, \mathbb{C}) \cong \bigoplus_{i=1}^{2} H^1(\mathbb{P}^1, L_{\xi^i}) \cong \mathbb{C} \oplus \mathbb{C}$$

.

We have another way to prove this claim.

Proof. Let $f : \chi \to S$ be universal family of $N \geq 4$ points in $\mathbb{P}^1$, $g : \mathcal{Z} \to S$ is the family of the d-th cyclic covers of $\mathbb{P}^1$ ramified on χ. Let $\pi : Z_s \to \mathbb{P}^1$ be any nonsingular fibre of the family, so Z_s is a d-cyclic cover of $\mathbb{P}^1$ ramified on $f^{-1}(s)$. Let $G =< \sigma >$ be the Galois group of the cover, so the image of the Kodaira-Spencer map $\mathrm{Kod} : \Theta_S \to R^1 g_* \Theta_{\mathcal{Z}/S}$ is the invariant part $H^1(Z_s, \Theta_{Z_s})^{inv}$ in $H^1(Z_s, \Theta_{Z_s})$ of the Galois group G, which is the logarithmic deformation of the pair $(\mathbb{P}^1, f^{-1}(s))$. So we have

$$H^1(Z_s, \Theta_{Z_s})^{inv} \cong H^1(\mathbb{P}^1, \Theta_{\mathbb{P}^1}(\log f^{-1}(s))).$$

So the image of Kod is isomorphism to $H^1(\mathbb{P}^1, \Theta_{\mathbb{P}^1}(\log f^{-1}(s)))$, then

$$\begin{aligned} & h^1(\mathbb{P}^1, \Theta_{\mathbb{P}^1}(\log f^{-1}(s))) \\ = \ & h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2-N)) \\ = \ & h^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-(2-N)-2)) \\ = \ & N-3 = \mu = \dim_{\mathbb{C}}(S) \end{aligned}$$

So the Kodaira-Spencer map is injective.

3.3 Infinity of CM Type Points

For a suitable choice of K, because of the claim 3.2.1 the family of Jacobians induces a generically finite morphism

$$\phi : S \longrightarrow {}_K M_{\mathbb{C}}(G, h_0).$$

3.3.1 ϕ Is Dominant

Proposition 3.3.1 *The maps $\phi : S \longrightarrow {}_K M_{\mathbb{C}}(G, h_0)$ are dominant for the following families*

$$\begin{aligned} y^d &= x(x-1)(x-\lambda_1), d = 4,5,6,7 \\ y^d &= x(x-1)(x-\lambda_1)(x-\lambda_2), d = 3,4,5 \\ y^3 &= x(x-1)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3), \\ y^2 &= x(x-1)\cdots(x-\lambda_\mu), \mu = 3,4. \end{aligned}$$

(The proof is similar to [16])

Proof. $d = 5,7, \mu = 1$ and $d = 3, \mu = 3$ are the cases of [5]. $d = 5, \mu = 2$ is the case of [16].

We first consider the case of $d = 6, \mu = 1$. Since $\dim S = 1$, and since ϕ is generically finite, we only need to show that

$$\dim({}_K M_{\mathbb{C}}(G, h_0)) = 1.$$

After base change we may assume that there exists a universal family

$$\pi : \mathcal{A} \longrightarrow {}_K M_{\mathbb{C}}(G, h_0),$$

together with an $\mathbb{Z}(\sigma)$-action on the fibres. This leads the eigenspaces decomposition of $R^1\pi_*(\mathbb{Q}_\mathcal{A})$ as polarized complex variation of Hodge structures

$$R^1\pi_*(\mathbb{Q}_\mathcal{A}) \otimes \mathbb{C} = \mathbb{V}(\xi) \oplus \mathbb{V}(\xi^2) \oplus \mathbb{V}(\xi^3) \oplus \mathbb{V}(\xi^4) \oplus \mathbb{V}(\xi^5).$$

Since $< lx, y > = < x, \bar{l}y >$, the intersection form $<,>$ induces a perfect duality between $\mathbb{V}(\xi^i)$ and $\mathbb{V}(\xi^{6-i})$.

Next we determine the ranks of the Hodge bundles in the corresponding decomposition. Note that the pull back of $R^1\pi_*(\mathbb{Q}_\mathcal{A})$ together with the $\mathbb{Z}(\sigma)$-action is just $R^1 g_*(\mathbb{Q}_{\mathrm{Jac}(\mathcal{Z})})$ together with the $\mathbb{Z}(\sigma)$-action. We only need to determine the ranks of the Hodge bundles in the decomposition

$$R^1 g_*(\mathbb{Q}_{\mathrm{Jac}(\mathcal{Z})}) \otimes \mathbb{C}) = \mathbb{W}(\xi) \oplus \mathbb{W}(\xi^2) \oplus \mathbb{W}(\xi^3) \oplus \mathbb{W}(\xi^4) \oplus \mathbb{W}(\xi^5).$$

Writing $h^{p,q}(\xi^i)$ for the rank of the (p,q) Hodge bundle of $\mathbb{W}(\xi^i)$, the perfect duality between $\mathbb{V}(\xi^i)$ and $\mathbb{V}(\xi^{6-i})$ induces

$$h^{1,0}(\mathbb{W}(\xi^i)) = h^{0,1}(\mathbb{W}(\xi^{6-i})).$$

We know σ acts on the holomorphic differentials of the fibre with eigenvalues ξ, ξ, ξ^2 and ξ^3. So one finds

$$\begin{aligned} (h^{1,0}(\xi), h^{0,1}(\xi)) &= (2,0), \quad (h^{1,0}(\xi^2), h^{0,1}(\xi^2)) = (1,0), \\ (h^{1,0}(\xi^3), h^{0,1}(\xi^3)) &= (1,1), \quad (h^{1,0}(\xi^4), h^{0,1}(\xi^4)) = (0,1), \\ (h^{1,0}(\xi^5), h^{0,1}(\xi^5)) &= (0,2). \end{aligned}$$

In particular, $\mathbb{W}(\xi)$, $\mathbb{W}(\xi^2)$, $\mathbb{W}(\xi^4)$, and $\mathbb{W}(\xi^5)$ are unitary local subsystems. The perfect duality between $\mathbb{V}(\xi^3)$ and $\mathbb{V}(\xi^3)$ implies that the corresponding the Higgs bundle

$$E^{1,0}(\xi^3) \longrightarrow E^{0,1}(\xi^3)$$

is dual to itself. This gives a precise description of the rank of the differential map

$$d : T_{{}_K M_{\mathbb{C}}(G,h_0)} \longrightarrow S^2E(0,1) \subset (E^{1,0})^{\otimes 2}$$

of the natural inclusion of ${}_K M_{\mathbb{C}}(G, h_0)$ into the moduli space of the polarized Abelian varieties in terms of the above eigenspaces decomposition. Since $\mathbb{V}(\xi)$, $\mathbb{V}(\xi^2)$, $\mathbb{V}(\xi^4)$, and $\mathbb{V}(\xi^5)$ are unitary, the differential map d factors over

$$d : T_{{}_K M_{\mathbb{C}}(G,h_0)} \longrightarrow (E^{1,0}(\xi^3) \oplus E^{1,0}(\xi^3))^{\vee} \otimes (E^{0,1}(\xi^3) \oplus E^{0,1}(\xi^3))$$

Since the Higgs field preserves the eigenspaces decomposition

$$(E^{1,0}(\xi^3)^{\vee} \oplus E^{0,1}(\xi^3)) \otimes (E^{0,1}(\xi^3) \oplus E^{1,0}(\xi^3)^{\vee}),$$

and d factors further through the diagonal map

$$\begin{aligned} d : T_{{}_K M_{\mathbb{C}}(G,h_0)} \longrightarrow E^{1,0}(\xi^3)^{\vee} \otimes E^{0,1}(\xi^3) &\oplus E^{1,0}(\xi^3)^{\vee} \otimes E^{0,1}(\xi^3)) \\ &\cong (E^{1,0}(\xi^3)^{\vee} \otimes E^{0,1}(\xi^3))^{\oplus 2}. \end{aligned}$$

The generical injectivity of d implies that the Kodaira-Spencer map on the each copy

$$\theta_{1,0} : T_{{}_K M_{\mathbb{C}}(G,h_0)} \longrightarrow E^{1,0}(\xi^3)^{\vee} \otimes E^{0,1}(\xi^3)$$

also is injective. Hence,

$$1 \leq \dim({}_K M_{\mathbb{C}}(G, h_0)) \leq \operatorname{rank}(E^{1,0}(\xi^3)^{\vee} \otimes E^{0,1}(\xi^3)) = 1.$$

Then we consider the case of $d = 3, \mu = 2$, the family of $y^3 = x(x-1)(x-\lambda_1)(x-\lambda_2)$. We have the decomposition

$$R^1g_*(\mathbb{Q}_{\mathrm{Jac}(\mathcal{Z})}) \otimes \mathbb{C}) = \mathbb{W}(\xi) \oplus \mathbb{W}(\xi^2).$$

And we have

$$(h^{1,0}(\xi), h^{0,1}(\xi)) = (2, 1), \qquad (h^{1,0}(\xi^2), h^{0,1}(\xi^2)) = (1, 2).$$

Hence,

$$1 \leq \dim({}_K M_{\mathbb{C}}(G, h_0)) \leq \operatorname{rank}(E^{1,0}(\xi^1)^\vee \otimes E^{0,1}(\xi^1)) = 2.$$

Then we consider the case of $d = 4$, $\mu = 1, 2$, which are the same up to a transform on the x-axis by

$$0 \mapsto 0, 1 \mapsto 1, \lambda_2 \mapsto \infty.$$

So we only need to prove the case for $d = 4$, $\mu = 1$, just the family of $y^4 = x(x-1)(x-\lambda_1)$. We have the decomposition

$$R^1 g_*(\mathbb{Q}_{\mathrm{Jac}(\mathcal{Z})}) \otimes \mathbb{C}) = \mathbb{W}(\xi) \oplus \mathbb{W}(\xi^2) \oplus \mathbb{W}(\xi^3)$$

And we have

$$(h^{1,0}(\xi), h^{0,1}(\xi)) = (2, 0), \qquad (h^{1,0}(\xi^2), h^{0,1}(\xi^2)) = (1, 1),$$
$$(h^{1,0}(\xi^3), h^{0,1}(\xi^3)) = (0, 2).$$

Hence,

$$1 \leq \dim({}_K M_{\mathbb{C}}(G, h_0)) \leq \operatorname{rank}(E^{1,0}(\xi^2)^\vee \otimes E^{0,1}(\xi^2)) = 1.$$

Now we only need to prove the case of $d = 2, \mu = 3, 4$. In fact $\mu = 3$ is the degenerate of $\mu = 4$, so we only need to prove the case of $d = 2, \mu = 3$, i.e. the family of $y^2 = x(x-1)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$.

First by claim 3.1.1 we have $\mathbb{Q} \hookrightarrow \mathrm{End}^0(\mathrm{Jac}(C_2^{(3)}))$. As above for some special $\lambda_1, \lambda_2, \lambda_3$, the curve is isomorphic to $y^2 = x^5 - 1$, then by a transformation $x \mapsto \frac{1}{x}$ the curve is just the case of theorem 4.2.1 6), so by [9] exercise 10.12 the Jacobian of this curve is simple, so the general member of Jacobians of this family is simple. Let F denote the endomorphism algebra of the general member, then we have

$$\mathbb{Q} \hookrightarrow F \hookrightarrow \mathbb{Q}(\xi),$$

so F must be a field. F can not be $\mathbb{Q}(\xi)$, otherwise the Jacobians of this family is rigid. If the dimension of F is two, then the general member can be: 1) general CM type with $e_0 = 1, d = 1, m = 2$, this can not be true because the universal family is of dimension 1; 2) real multiplication with $e_0 = 2, m = 1$, this can not be true because the universal family is of dimension 2. So $F = \mathbb{Q}$, the general member is of real multiplication with endomorphism algebra $\mathbb{Q}$, and the universal family is of dimension $\frac{m(m+1)}{2} = \frac{2\times(2+1)}{2} = 3$.

3.3.2 Infinity of CM Points

Theorem 3.3.1 *There are infinite many CM type points in the following families of Jacobians :*

$$\begin{aligned} y^d &= x(x-1)(x-\lambda_1), d = 4,5,6,7 \\ y^d &= x(x-1)(x-\lambda_1)(x-\lambda_2), d = 3,5 \\ y^d &= x(x-1)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3), d = 2,3 \end{aligned}$$

Proof. By [11] section 2, the points of ${}_K M_{\mathbb{C}(G,h_0)}$ that correspond to abelian varieties that are of CM type are exactly the points that are in the image of a map

$$M_{\mathbb{C}}(H,h') \to M_{\mathbb{C}}(G,h_0) \to {}_K M_{\mathbb{C}}(G,h_0)$$

for some torus $u : H \hookrightarrow G$ and some map h' as in [3] 3.13. Combining [3] 5.1 and 5.2 we see that the set of points obtained in this way is dense. So the set of $\lambda = (\lambda_1, \ldots, \lambda_\mu) \in S$ for which $\mathrm{Jac}(C_\lambda)$ is of CM type is dense.

By the proposition 3.3.1, $\phi : S \to {}_K M_{\mathbb{C}}(G,h_0)$ is dominant for the above families of curves, so there are infinite many CM type points in the above families.

Chapter 4

Endomorphism Algebras of Jacobians of Some Families of Curves

Before we discuss the endomorphism algebras of Jacobians of families of curves, we will prove some facts about the relations of the Jacobians between tow curves C and C', where C is a finite covering of C'. Here we follow as [9].

4.1 Jacobians Between Finite Coverings of Curves

Let $f : C \to C'$ be a finite morphism between two smooth projective curves, we will consider the Jacobians between these two curves. Denote J the Jacobian of the curve C, we have

Proposition 4.1.1 *The homomorphism $f^* : J' \to J$ is not injective if and only* *if f factorizes via a cyclic étale covering f' of degree $n \geq 2$:*

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[d, "f''"'] & C' \\ C'' \arrow[r, equal] & C'' \arrow[u, "f'"'] \end{tikzcd}$$

Proof. Suppose first that f factorizes via a cyclic étale covering f' of degree $n \geq 2$. It suffices to show that the homomorphism $f'^* : J' \to J'' = J(C'')$ is not injective. To see this, recall that f' is given as follows: there exists a line bundle L on C' of order n in $\mathrm{Pic}^0(C')$ such that C'' is the inverse image of the unit section of $L^n = C' \times \mathbb{C}$ under the n-th power map $L \to L^n$ and $f' : C'' \to C'$ is the restriction of $L \to L^n$ to C''. Denote by $p : L \to C'$ the natural projection. Since the tautological line bundle p^*L is trivial, so is $f'^*L = p^*L \mid C''$, and thus f'^* is not injective.

Conversely, suppose f^* is not injective. Choose a nontrivial line bundle $L \in \ker f^* \subset \mathrm{Pic}^0(C')$. Necessarily L is of finite order, say $n \geq 2$, since

$$L^{\deg f} = N_f f^* L = N_f \mathcal{O}_C = \mathcal{O}_{C'}.$$

Then the cyclic étale covering $f' : C'' \to C'$ associated to L is of degree n. Consider the pullback diagram

$$\begin{tikzcd} C \times_{C'} C'' \arrow[r, "q"] \arrow[d, "p"'] & C'' \\ C \arrow[r, "f"'] & C'. \arrow[u, "f'"'] \end{tikzcd}$$

The étale covering p is given by the trivial line bundle $f^*L = \mathcal{O}_C$. Hence $C \times_{C'} C''$ is the disjoint union of n copies of C. In particular there exists a section $s : C \to C \times_{C'} C''$ and f factorizes as $f = f'qs$.

From the proof of the Proposition one easily deduces that for the cyclic étale covering $f' : C'' \to c'$ the kernel $\ker\{f^* : J' \to J''\}$ is generated by the line bundle L defining f'. If $(f'')^* : J'' \to J$ is not injective, one can apply the proposition again and factorize f''. Repeating this process we obtain

Corollary 4.1.1 *For any finite morphism $f : C \to C'$ of smooth projective curves C and C' there is a factorization*

$$\begin{array}{ccc} C & \xrightarrow{f} & C' \\ g\downarrow & & \uparrow f_e \\ C_e & = & C_e \end{array}$$

with f_e étale, $kerf^* = kerf_e^*$, and $g^* : J(C_e) \to J$ injective.

4.2 Endomorphism Algebras of Families of Jacobians

Now we discuss the endomorphism algebra of a general (general has the same meaning as in section 2.4) member of the families $\mathrm{Jac}(\mathcal{Z}/S)$. If $p \mid d$, then $y^d = x(x-1)(x-\lambda_1)\dots(x-\lambda_\mu)$ (which we denote as $C_d^{(\mu)}$) is a cyclic covering of $y^p = x(x-1)(x-\lambda_1)\dots(x-\lambda_\mu)$ of degree $\frac{d}{p}$ not factorizes via a cyclic étale covering, so by corollary 4.1.1 we have

$$\mathrm{Jac}(C_p^{(\mu)}) \hookrightarrow \mathrm{Jac}(C_d^{(\mu)}).$$

4.2.1 The Case For $\mu = 1$

Claim 4.2.1 *The general member of Jacobians of $y^5 = x(x-1)(x-\lambda_1)$ is simple, and the endomorphism algebra of the general member is $\mathbb{Q}(\xi)$, where ξ is a primitive root of $t^5 - 1 = 0$.*

Proof. We have

$$\mathbb{Q}(\xi) \hookrightarrow \mathrm{End}^0(\mathrm{Jac}(C_5^{(1)}))$$

by the claim 3.1.1.

Take a transformation determined by

$$0 \mapsto e^{\frac{2\sqrt{-1}\,\pi}{3}}, 1 \mapsto 1, \infty \mapsto \infty,$$

the family becomes $y^p = (x - e^{\frac{2\sqrt{-1}\,\pi}{3}})(x-1)(x-\lambda_1')$. For $\lambda_0 \mapsto \lambda_0' = e^{\frac{4\sqrt{-1}\,\pi}{3}}$, the curve is isomorphic to

$$y^5 = x^3 - 1$$

by corollary 3.1.1, we have $\mathbb{Q}(\zeta) \hookrightarrow \mathrm{End}^0(\mathrm{Jac}(Z_{\lambda_0}))$, where ζ is a primitive root of $t^{15} - 1 - 0$, then $\mathrm{Jac}(y^5 = x^3 - 1)$ is a simple abelian variety of CM type with endomorphism algebra $\mathbb{Q}(\zeta)$ by [5] 2.17.

If F is the endomorphism algebra of the general member of this family, then

$$\mathbb{Q}(\xi) \subset F \subset \mathbb{Q}(\zeta),$$

then F is field, so the general member of this family is simple. F can not be $\mathbb{Q}(\zeta)$, otherwise the family is rigid. But $\dim_{\mathbb{Q}(\xi)} \mathbb{Q}(\zeta) = 2$, so $F = \mathbb{Q}(\xi)$.

Claim 4.2.2 *The family $Jac(C_4^{(1)})$ is isogenous to $Jac(C_2^{(1)}) \times E_1^2$, where E_1 is an elliptic curve of CM type. So the endomorphism algebra of the general member of this family is $\mathbb{Q} \oplus M_2(\mathbb{Q}(\sqrt{-1}))$.*

Proof. $C_4^{(1)} \longrightarrow C_2^{(1)}$ induces $\mathrm{Jac}(C_2^{(1)}) \hookrightarrow \mathrm{Jac}(C_4^{(1)})$, so $\mathrm{Jac}(C_4^{(1)}) \sim \mathrm{Jac}(C_2^{(1)}) \times E_2$ ($\sim$ means isogenous), where E_2 is a family of abelian varieties of dimension 2.

By claim 3.1.1 we have the decomposition

$$\begin{aligned} R^1 g_*(\mathbb{Q}_{\mathrm{Jac}(\mathcal{Z}/S)}) \otimes \mathbb{C}) &= \mathbb{W}(\sqrt{-1}) \oplus \mathbb{W}(-1) \oplus \mathbb{W}(-\sqrt{-1}) \\ &= (\mathbb{W}(\sqrt{-1}) \oplus \mathbb{W}(-\sqrt{-1})) \oplus (\mathbb{W}(-1)). \end{aligned}$$

$\mathbb{W}(-1)$ is defined over $\mathbb{Q}$, so $\mathbb{W}(\sqrt{-1}) \oplus \mathbb{W}(-\sqrt{-1})$ is defined over $\mathbb{Q}$ too, so E_2 is the abelian variety determined by $\mathbb{W}(\sqrt{-1}) \oplus \mathbb{W}(-\sqrt{-1})$, so E_2 is independent of λ_1 (because the eigenvalues are $\sqrt{(-1)}, \sqrt{(-1)}$ and -1), that means E_2 is rigid. By considering the semi-stable reduction at $\lambda_1 = 0$, we know that the singular fibre of $\mathrm{Jac}(C_4^{(\mu)})$ is $\mathbb{C}^* \times (\mathrm{Jac}(y^4 = x^2 - 1))^2$, so E_2 is just $(\mathrm{Jac}(y^4 = x^2 - 1))^2$. It is not difficult to see the endomorphism algebra of $\mathrm{Jac}(y^4 = x^2 - 1)$ is $\mathbb{Q}(\sqrt{-1})$.

Claim 4.2.3 *The family $Jac(C_6^{(1)})$ is isogenous to $Jac(C_2^{(1)}) \times E_1^3$, where E_1 is an elliptic curve of CM type, and $End^0(E_1)$ is $\mathbb{Q}(\xi)$, where ξ is a primitive root of $t^3 - 1 = 0$.*

Proof. $C_6^{(1)} \longrightarrow C_2^{(1)}$ and $C_6^{(1)} \longrightarrow C_3^{(1)}$ induces $\mathrm{Jac}(C_2^{(1)}) \hookrightarrow \mathrm{Jac}(C_6^{(1)})$ and $\mathrm{Jac}(C_3^{(1)}) \hookrightarrow \mathrm{Jac}(C_6^{(1)})$ respectively, so $\mathrm{Jac}(C_6^{(1)}) \sim \mathrm{Jac}(C_2^{(1)}) \times E_1 \times E_2$, where $E_1 = \mathrm{Jac}(C_3^{(1)})$ is an elliptic curve of CM type and E_2 is a rigid abelian surface because the decomposition

$$\begin{aligned} R^1 g_*(\mathbb{Q}_{\mathrm{Jac}(Z/S)}) \otimes \mathbb{C}) &= \mathbb{W}(\xi) \oplus \mathbb{W}(\xi^2) \oplus \mathbb{W}(\xi^3) \oplus \mathbb{W}(\xi^4) \oplus \mathbb{W}(\xi^5) \\ &= \mathbb{W}(\xi^3) \oplus \left(\mathbb{W}(\xi^2) \oplus \mathbb{W}(\xi^4)\right) \oplus \left(\mathbb{W}(\xi) \oplus \mathbb{W}(\xi^5)\right) \end{aligned}$$

is defined over $\mathbb{Q}$, so E_1 and E_2 are independent of λ_1 (because the eigenvalues are ξ, ξ, ξ^2 and ξ^3). In order to see wether E_2 is split, we consider the semi-stable reduction at $\lambda_1 = 0$. After semi-stable reduction the Jacobian of the curve is $\mathbb{C}^* \times E_1 \times E_2$, where E_2 is the Jacobian of the curve which is 6-cyclic cover of $\mathbb{P}^1$ ramified at three points, among them tow points are total ramified, so E_2 is isomorphic to the Jacobian of the normalization of

$$y^6 = x^2 - 1.$$

By [9] exercise 10.12 we know that $\mathrm{Jac}(y^6 = x^2 - 1)$ is isomorphic to a product of elliptic curves, and the endomorphism algebra of $\mathrm{Jac}(y^6 = x^2 - 1)$ is not commutative (because the automorphism group of this curve action on holomorphic forms is not commutative), so E_2 is isomorphic to $(E_1')^2$, where E_1' is an elliptic curve of CM type (because E_2 is rigid). But $y^6 = x^2 - 1$ factorizes via $y^3 = x^2 - 1$, so E_1' must be E_1.

4.2.2 The Case For $\mu = 2$

Claim 4.2.4 *The general member of Jacobians of the family $y^3 = x(x-1)(x-\lambda_1)(x-\lambda_2)$ is simple, and the endomorphism algebra of the general member is $\mathbb{Q}(\xi)$, where ξ is a primitive root of $t^3 - 1 = 0$.*

Proof. By the claim we have $\mathbb{Q}(\xi) \hookrightarrow \mathrm{End}^0(\mathrm{Jac}C_3^{(2)})$. After take a transformation and take some special λ_1, λ_2 we have a curve $y^3 = x(x^3-1)$, by the corollary 3.1.2 we have $\mathbb{Q}[t]/(t^6+t^3+1) \hookrightarrow \mathrm{End}^0(\mathrm{Jac}(y^3 = x(x^3-1)))$, so $\mathrm{Jac}(y^3 = x(x^3-1))$ can only be simple or isogenous to E_1^3, otherwise the endomorphism algebra cannot contain a field of dimension 6. After semi-stable reduction of the family at $\lambda_1 = \lambda_2 = 0$, the Jacobian is $(\mathbb{C}^*)^2 \times E_1$, so it cannot be isogenous to E_1^3, so $\mathrm{Jac}(y^3 = x(x^3-1))$ is a simple abelian variety, so the general member of the family is simple with endomorphism algebra $\mathbb{Q}(\xi)$.

Theorem 4.2.1 (Bolza, [1]) *Let C be a smooth projective curve of genus 2 with nontrivial reduced automorphism group. Then C is isomorphic to one of the 6 types of curves in the following:*

1) $y^2 = (x^2-a^2)(x^2-b^2)(x^2-1)$ and $\overline{Aut\ C} \cong \mathbb{Z}/2$;

2) $y^2 = x(x^2-a^2)(x^2-a^{-2})$ and $\overline{Aut\ C} \cong D_2$;

3) $y^2 = (x^3-a^3)(x^3-a^{-3})$ and $\overline{Aut\ C} \cong D_3$;

4) $y^2 = x^6-1$ and $\overline{Aut\ C} \cong D_6$;

5) $y^2 = x(x^4-1)$ and $\overline{Aut\ C} \cong \mathfrak{S}_4$;

6) $y^2 = x(x^5-1)$ and $\overline{Aut\ C} \cong \mathbb{Z}/5$;

Where $\overline{Aut\ C}$ denotes the reduced automorphism group of C, that is $Aut\ C$ modulo the hyperelliptic involution.

The Jacobians of the family $y^3 = x^2(x-1)(x-\lambda_1)$ is the bound of the family $y^3 = x(x-1)(x-\lambda_1)(x-\lambda_2)$, which is also a Shimura curve of abelian varieties. Denote $\mathcal{Y}$ be the family of Jacobians of the normalization of $y^3 = x^2(x-1)(x-\lambda_1)$, so $\mathcal{Y}$ is a family of abelian surfaces.

Claim 4.2.5 *The family $\mathcal{Y}$ is isomorphic to a product of families of elliptic curves $\mathcal{E}_1^2$, the endomorphism algebra of a general member of $\mathcal{E}_1$ is $\mathbb{Q}$.*

Proof. By claim 3.1.1, $\mathbb{Q}(\xi) \hookrightarrow \mathrm{End}^0(Y_{\lambda_1})$, where ξ is a primitive root of $t^3 - 1 = 0$, so $y^3 = x^2(x-1)(x-\lambda_1)$ is a curve of genus 2 with nontrivial reduced automorphism group, so it must be some case of theorem 4.2.1. So if the general member of $\mathcal{Y}$ is simple, then it can be the following cases:

a) of totally indefinite quaternion multiplication.

b) of totally definite quaternion multiplication.

c) of general CM type with endomorphism algebra $\mathbb{Q}(\xi)$.

If a), $\mathcal{Y}$ is a compact Shimura curve by [14], contradiction to the fibre at $\lambda_1 = 0$ is $(\mathbb{C}^*)^2$ (just the case 2) of theorem 4.2.1). If b), $\mathcal{Y}$ is a rigid family of product of an elliptic curve with CM type with itself ([9] example 9.5.5). If c), then $\mathcal{Y}$ cannot be of the case: 1) of theorem 4.2.1, otherwise the general member of $\mathcal{Y}$ is isogenous to a product of elliptic curves ([9] exercise 10.12); 2) of theorem 4.2.1, because the reduced automorphism group D_2 cannot contain a subgroup of order 3; 3), 4), 5) of theorem 4.2.1, otherwise it is isomorphic to a product of elliptic curves; 6) of theorem 4.2.1, otherwise it is rigid. So the general member of $\mathcal{Y}$ is not simple. Because the fibre at $\lambda_1 = 0$ is $(\mathbb{C}^*)^2$, so the general member of $\mathcal{Y}$ cannot contain a copy of CM type, so $\mathcal{Y}$ is isomorphic to a product of families of elliptic curves $\mathcal{E}_1 \times \mathcal{E}_2$, if $\mathcal{E}_1$ and $\mathcal{E}_2$ are not the same, then the endomorphism algebra of a general member of $\mathcal{E}_1 \times \mathcal{E}_2$ is $\mathbb{Q} \oplus \mathbb{Q}$, this cannot contain a subfield of $\mathbb{Q}(\xi)$, contradiction arises. So $\mathcal{Y}$ is isomorphic to a product of family of elliptic curves $\mathcal{E}_1^2$.

Claim 4.2.6 *The family $Jac(C_6^{(2)})$ is isogenous to $Jac(C_2^{(1)}) \times Jac(C_3^{(2)}) \times (E_1)^3$, where E_1 is an elliptic curve of CM type with endomorphism algebra $\mathbb{Q}(\xi)$, where ξ is a primitive root of $t^3 - 1 = 0$.*

Proof. Obviously $\mathrm{Jac}(C_6^{(2)})$ is isogenous to $\mathrm{Jac}(C_2^{(1)}) \times \mathrm{Jac}(C_3^{(2)}) \times \mathcal{E}_3$, where $\mathcal{E}_3$ is a family of abelian varieties of dimension 3. The following decomposition

$$R^1 g_*(\mathbb{Q}_{\mathrm{Jac}(\mathcal{Z}/S)}) \otimes \mathbb{C}) \;=\; \mathbb{W}(\xi^3) \oplus \big(\mathbb{W}(\xi^2) \oplus \mathbb{W}(\xi^4)\big) \oplus \big(\mathbb{W}(\xi) \oplus \mathbb{W}(\xi^5)\big)$$

is defined over $\mathbb{Q}$ and $(h^{1,0}, h^{01}) = (3, 0)$, so $\mathcal{E}_3$ is a rigid family of E_3. The Jacobian of the semi-stable reduction at $\lambda_1 = \lambda_2 = 0$ is $(\mathbb{C}^*)^2 \times \mathrm{Jac}(C_3^{(1)}) \times E_4$, where E_4 is the Jacobian of $y^6 = x^3 - 1$. By claim 4.2.3 we know E_4 is isogenous to $E_1 \times E_1^3 = E_1^4$, so E_3 is isogenous to E_1^3, where E_1 is an elliptic curve of CM type.

4.2.3 The Case For $\mu = 3$

Claim 4.2.7 *The general member of Jacobians of the family $y^2 = x(x-1)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$ is simple, and the endomorphism algebra of the general member is $\mathbb{Q}$.*

Proof. This is just the case $d = 2$ and $\mu = 3$ of proposition 3.3.1.

Claim 4.2.8 *The general member of Jacobians of the family $y^3 = x(x-1)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3)$ is simple, and the endomorphism algebra of the general member is $\mathbb{Q}(\xi)$, where ξ is a primitive root of $t^3 - 1 = 0$.*

Proof. By the claim 3.1.1 we have $\mathbb{Q}(\xi) \hookrightarrow \mathrm{End}^0(\mathrm{Jac}(C_3^{(3)}))$. After take a transformation and take some special $\lambda_1, \lambda_2, \lambda_3$ we have a curve $y^3 = x^5 - 1$, by [5] 2.17 $\mathrm{Jac}(y^3 = x^5 - 1))$ is simple with endomorphism algebra $\mathbb{Q}(\zeta)$, where ζ is a primitive root of $t^{15} - 1 = 0$. So the general member of this family is simple. Let F be the endomorphism algebra of general member, then $\mathbb{Q}(\xi) \subset F \subset \mathbb{Q}(\zeta)$, then F must be $\mathbb{Q}(\xi)$ otherwise the dimension of the family would less than 3.

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